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A BRIEF ACCOUNT

OF THE

HISTORICAL DEVELOPMENT OF PSEUDOSPHERICAL SURFACES FROM 1827 TO 1887.

Submitted in partial fulfilment of the requirements for the degree of Doctor of Philosophy, in the Faculty of Pure Science, Columbia University.

BY

EMILY CODDINGTON.

Press of
The New Era Printing Company,
Lancaster, Pa.
1905

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I

THE APPLICATION OF ONE PSEUDOSPHERICAL SURFACE UPON ANOTHER AND THE GEOMETRY OF THESE SURFACES.

1. The definition of pseudospherical surfaces.
2. The definition of total curvature according to Gauss.
3. The application of surfaces with constant curvature upon one another.
4. The pseudospherical surfaces of rotation and the helicoidal surface.
5. Enneper's surfaces.
6. Asymptotic lines on a pseudospherical surface.
7. The identification of the non-Euclidean geometry with pseudospherical geometry.
8. The projections of a pseudospherical surface upon a plane analogous to the central and stereographic projection of a sphere on a plane.
9. The identification of pseudospherical geometry with the metrical geometry of Cayley.

II.

THE SURFACE OF CENTERS AND THE TRANSFORMATION OF ONE PSEUDOSPHERICAL SURFACE INTO ANOTHER.

1. The theorem of transformation.
2. Kummer's theory of congruences.
3. Weingarten's two theorems on surfaces whose radii of curvature are functionally related.
4. Ribaucour's cyclic system of surfaces.
5. Bianchi's complementary transformation.
6. Geodesic lines on the surface of centers.
7. Lie's transformation.
8. Bäcklund's transformation.
9. Darboux's equations for Bianchi's and Bäcklund's transformation.
10. The triply orthogonal system of surfaces.

BOOKS OF REFERENCE.

1. **Dupin, C.**, Développements de géométrie. Paris, 1821.
2. **Gauss, J.**, Disquisitiones generales circa surperficies curvas. 1827. Translated into English by Messrs. Morehead and Hiltebeitel.
3. **Minding, E. F. A.**, Bemerkung über die Abwickelung krummer Linien von Flächen. J. für Math., VI., 1830.
4. **Lobatschewsky**, Kasaner Boton, 1829. Neue Anfangsgrunde der Geometrie nebst einer vollstandigen Theorie der Parallelen. Kasan, 1836–1836. Translated into French by J. Hoüel.

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5. **Minding, E. F. A.**, Ueber die Biegung gewisser Flächen. J. für Math., XVIII., 1838.
6. ——— Wie sich entscheiden lässt, ob zwei gegebene krumme Flächen auf einander abwickelbar sind oder nicht; nebst Bemerkungen über die Flächen von unveränderlichem Krümmungsmasse. J. für Math., XIX., 1839.
7. ——— Zur Theorie der kürzesten Linien auf krummen Flächen. J. für Math., XX., 1840.
8. **Serret, J. A.**, Note sur une équation aux dérivées partielles. Journ. de Math., XIII., 1848.
9. **Bonnet, O.**, Mémoire sur la théorie générale des surfaces. J. de l'Éc. Pol. cah., XXXII., 1848.
10. **Liouville, J.**, IV. Note to Application de l'analyse a la géometrie, by G. Monge. 1850.
11. **Liouville, J.**, Sur l'équation aux différences partielles, $\frac{\partial^2 \log \lambda}{\partial u \partial v} \pm \frac{\lambda}{2u^2} = 0$. Journ. de Math., XVIII., 1853.
12. **Jellet, J. H.**, On the Properties of the Inextensible Surfaces. Transactions of the Irish Academy, Vol. 22, 1853.
13. **Riemann, B.**, Ueber die Hypothesen, welche der Geometrie zu Grunde liegen. Gött. Abh., 1867, written in 1854, translated into English by W. K. Clifford, Nature, Vol. VII.
14. **Cayley, A.**, Sixth Memoir upon Quantics. Philosoph. Transactions, Band 149, 1859.
15. **Codazzi**, Nota intorno le superficie, che hanno costante il prodotto dei due raggi di curvatura. Annali di Tortolini, T. VIII., 1857.
16. **Kummer**, Allgemeine Theorie der geradlinigen Strahlensysteme. Journal von Crelle, 57, 1860.
17. **Weingarten, J.**, Ueber eine Klasse auf einander abwickelbarer Flächen. J. für Math., LIX., 1861.
18. **Enneper, A.**, Ueber einige Formeln aus der analytischen Geometrie der Flächen. Schlömilch Z., VII., 1862.
19. **Bour, O.**, Théorie de la déformation des surfaces. J. de l'Éc. Pol. cah. XXXIX., 1862.
20. **Weingarten, J.**, Ueber die Oberflachen, für welche einer der beiden Hauptkrümmungshalbmesser eine Function des anderen ist. J. für Math., LXII., 1863.
21. **Beltrami, E.**, Intorno ad alcune proprietà delle superficie di rivoluzione. Annali di Mat., VI., 1864.
22. **Dini, U.**, Sulla teoria della superficie. Batt. G., III., 1865.
23. ——— Sulla superficie gobbe che possono, etc. Batt. G., III., 1865.
24. ——— Sulle superficie di curvatura costante, Batt. G., III., 1865.
25. ——— Des surfaces à courbure costante. C. R., LX., 1865.
26. **Beltrami, E.**, Ricerche di analisi applicata alla geometria. Batt. G., III., 1865.
27. ——— Dimostrazione di due formole del signor Bonnet. Batt. G., IV., 1866.
28. ——— Risoluzione del problema; Riportare i punti di una superficie sopra un piano in modo, che le linee geodetiche vengano rappresentate da linee rette. Annali di Mat., VII., 1865.
29. **Dini, U.**, Sopra alcuni punti della teoria delle superficie applicabili. Annali di Mat., VII., 1866.
30. **Helmholtz**, Ueber die thatsachlichen Grundlagen der Geometrie. 1866.

31. **Bonnet, O.**, Mémoire sur la théorie des surfaces applicables sur une surface donnée. J. de l'Éc. Pol. cah., XLII., 1867.

32. **Beltrami, E.**, Saggio d'interpretazione della geometria noneuclidea. Batt. G., VI., 1868. Translated into French by J. Houël, Annales de l'École Normale, VI.

33. **Christoffel, E. B.**, Allgemeine Theorie der geodätischen Dreiecke. Berl. Abh., 1868.

34. **Dini, U.**, Sopra alcuni punti della teoria delle superficie. Mem. della società italiana (3), I., 1868.

35. **Enneper, A.**, Analytisch-geometrische Untersuchungen. Gött. Nachr., 1868.

36. ——— Die cyklischen Flächen. Schlömilch Z., XIV., 1869.

37. **Beltrami, E.**, Teoria fondamentale degli spazii di curvatura costante. Annali di Mat. (2), II., 1869.

38. **Dini, U.**, Ricerche sopra la teoria delle superficie. Mem. della società italiana (3), II., 1869.

39. ——— Sulle superficie, che hanno un sistema di linee di curvatura piane. Ann. d. Univ. Toscana, 1869.

40. **Enneper, A.**, Ueber eine Erweiterung des Begriffs von Parallel flächen. Gött. Nachr., 1870.

41. ——— Ueber asymptotische Linien. Gött. Nachr., 1870.

42. ——— Untersuchungen über einige Punkte aus der allgemeinen Theorie der Flächen. Math. Ann., II., 1870.

43. **Ribaucour, A.**, Sur la déformation des surfaces. C. R., LXX., 1870.

44. **Dini, U.**, Sulle superficie, che hanno un sistema di linee di curvatura sferiche. Mem. della società italiana (3), II., 1870.

45. **Klein, F.**, Ueber die sogenannte Nicht-Euklidische Geometrie. Math. Ann., IV., 1871.

46. **Kretschmer, E.**, Zur Theorie der Flächen mit ebenen Krümmungslinien. Progr. Frankfurt a. O., 1871.

47. **Dini, U.**, Sopra alcune formole generali della teoria delle superficie. Annali di Mat. (2), IV., 1871.

48. **Beltrami, E.**, Sulla superficie di rotazione, che serve di tipo alle superficie pseudosferiche. Batt. G., X., 1872.

49. ——— Teorema di geometria pseudosferica. Batt. G., X., 1872.

50. **Ribaucour, A.**, Sur les développées des surfaces. C. R., LXXIV., 1872.

51. **Ribaucour, A.**, Sur les systêmes cycliques. C. R., LXXVI., 1873.

52. **Enneper, A.**, Bemerkungen über geodätische Linien. Schlömilch. Z., XVIII., 1873.

53. **Escherich, G. v.**, Die Geometrie auf den Flächen constanter negativer Krümmung. Wien. Ber., LXIX., 1874.

54. **Enneper, A.**, Bemerkung zu den analytisch-geometrischen Untersuchungen. Gött. Nachr., 1874.

55. **Klein, F.**, Ueber Nicht-Euklidische Geometrie. Math. Ann., VII., 1874.

56. **Simon, P.**, Ueber Flächen mit constantem Krümmungsmass. Diss. Halle, 1876.

57. **Enneper, A.**, Ueber einige Flächen mit constantem Krümmungsmass. Gött. Nachr., 1876.

58. **Bockwoldt, G.**, Ueber die Enneperchen Flächen mit constantem positivem Krümmungsmass. Diss. Göttingen, 1876.

59. **Hazzadakis, J. N.**, Ueber einige Eigenschaften der Flachen mit constantem Krümmungsmass. J. für Math., LXXXVIII., 1879.
60. **Lie, S.**, Zur Theorie der Flächen constanter Krümmung. Arch. for Mat. og. Naturv., IV., 1879.
61. ——— Ueber Flächen, deren Krümmungsradien durch eine Relation verknüpft sind. Arch. for Mat. og. Naturv., IV., 1879.
62. ——— Klassifikation der Flächen nach der Transformationsgruppe ihrer geodätischen Curven. Universitatsprogr. Christiania, 1879.
63. **Bianchi, L.**, Ricerche sulle superficie a curvatura costante e sulle elicoidi. Annali di Pisa, II., 1879.
64. **Cayley, A.**, On the correspondence of homographies and rotations. Math. Ann., XV., 1879.
65. **Lecornu**, Sur l'équilibre des surfaces flexibles et inextensibles. J. de l'Éc. Poly., XXIX., 1880.
66. **Lenz**, Ueber die Enneperschen Flächen constanter negativer Krümmung. Diss. Göttingen, 1879.
67. **Lie, S.**, Sur les surfaces, dont les rayons de courbure ont entre eux une relation. Darboux Bull. (2), IV., 1880.
68. ——— Zur Theorie der Flächen constanter Krümmung. Arch. for Mat. og Naturv., V., 1880.
69. **Bianchi, L.**, Ueber der Flächen mit constanter negativer Krümmung. Math. Ann., XVI., 1880.
70. **Bäcklund, A. V.**, Zur Theorie der partiellen Differentialgleichungen. Math. Ann., XVII., 1880.
71. **Voss, A.**, Ueber ein Princip der Abbildung krummer Oberflächen. Math. Ann., XIX., 1881.
72. **Nebelung**, Trigonometrie der Flächen mit constantem Krümmungsmass. Pr. Dortmund, 1881.
73. **Lie, S.**, Transformationstheorie der partiellen Differentialgleichung.

$$s^2 - rt = \frac{(1 + p^2 + q^2)^2}{a^2}.$$

Arch. for Mat. og. Naturv., VI., 1881.
74. **Lie, S.**, Diskussion der Differentialgleichung $s = F(z)$. Arch. for Mat. og. Naturv., VI., 1881.
75. **Bäcklund, A. V.**, Zur Theorie der Flächentransformationen. Math. Ann., XIX., 1881.
76. **Haas, A.**, Vorsuch einer Darstellung der Geschichte des Krümmungsmasses. 1881.
77. **Bianchi, L.**, Sulle superficie a curvatura costante positiva. Batt. G., XX., 1882.
78. **Weingarten, J.**, Ueber die Verschiebbarkeit geodätischer Dreiecke in krummen Flächen. Berl. Ber., 1882.
79. ——— Ueber die Eigenschaften des Linienelementes der Flächen von constantem Krümmungsmass. J. für Math., XCIV. u. XCV., 1883.
80. **Darboux, G.**, Sur les surfaces, dont la courbure totale est constante. C. R., XCVII., 1883.
81. ——— Sur l'équation aux dérivées partielles des surfaces à courbure constante. C. R., XCVII., 1883.
82. **Mangoldt, H. v.**, Klassifikation der Flächen nach der Verschiebbarkeit ihrer geodätischen Dreiecke. J. für Math., XCIV., 1883.

83. **Bianchi, L.**, Sopra alcune classi di sistema tripli ciclici di superficie ortogonali. Batt. G., XXI., 1883.
84. **Lie, S.**, Untersuchungen über Differentialgleichungen. Christiania Forh., XVIII., 1883.
85. **Kuen, Th.**, Ueber Flächen von constantem Krümmungsmass. Münch. Ber., 1884.
86. **Bäcklund, A. V.**, Om ytor med konstant negativ krökning. Lund. Årsskr., 1884.
87. **Cayley, A.**, On the non-Euclidian plane geometry. London R. Soc. Proceed., XXXVII., 1884.
88. **Bianchi, L.**, Sui sistemi tripli ciclici di superficie ortogonali. Batt. G., XXII., 1884.
89. ——— Sopra una classe di sistemi tripli di superficie ortogonali. Annali di Mat. (2), XIII., 1885.
90. ——— Sopra i sistemi tripli ortogonali di Weingarten. Annali di Mat. (2), XIII., 1885.
91. ——— Sopra i sistemi tripli ortogonali di Weingarten. Rom. Acc. L. Rend. (4), I., 1885.
92. **Volterra, V.**, Sulla deformazione delle superficie flessibili ed inestendibili. Rom. Acc. L. Rend. (4), I., 1885.
93. **Bianchi, L.**, Aggiunte alla memoria; "Sopra i sistemi tripli ortogonali di Weingarten." Annali di Mat. (2), XIV., 1886.
94. ——— Sopra i sistemi tripli ortogonali, che contengono un sistema di superficie pseudosferische. Rom. Acc. L. Rend. (4), II., 1886.
95. **Weingarten, J.**, Ueber die Deformationen einer biegsamen unausdehnbaren Fläche. J. für Math., Vol. C., 1886.
96. **Dobriner, H.**, Die Flächen constanter Krümmung mit einem System sphärischer Krümmungslinien. Acta Math., IX., 1886.
97. **Lipschitz, R.**, Ueber die Oberflächen, bei denen die Differenz der Hauptkrümmungsradien constant ist. Acta Math., X., 1887.
98. **Bianchi, L.**, Sopra i sistemi doppiamente infiniti di raggi. Rom. Acc. L. Rend. (4), III., 1887.
99. ——— Sui sistemi doppiamente infiniti di raggi. Annali di Mat. (2), XV., 1887.
100. **Oekinghaus, E.**, Ueber die Pseudosphäre. Hoppe Arch. (2), V., 1887.
101. **Lilienthal, R. v.**, Zur Theorie der Krümmungsmittelpunktsflächen. Math. Ann., XXX., 1887.
102. ——— Bemerkung uber diejenigen Flächen, bei denen die Differenz der Hauptkrümmungen constant ist. Acta Math., XI., 1888.
103. **Nannei, E.**, Le superficie iperciclîche. Napoli Rend. (2), II., 1888.
104. ——— Le superficie ipercicliche. Batt. G., XXVI., 1888.
105. **Marcolongo, R.**, Sulla rappresentazione conforme della pseudofera e sue applicazioni. Napoli Rend. (2), II., 1888.
106. **Razzaboni, A.**, Sopra certe famiglie di superficie di rivoluzione applicabili. Bologna Acc. Rend., 1888.
107. **Pirondini, G.**, Studio sulle superficie elicoidali. Annali di Mat. (2), XVI., 1888.
108. **Weingarten, J.**, Ueber eine Eigenschaft der Flächen, bei denen der eine Hauptkrümmungsradius eine Function des anderen ist. J. für Math., CIII., 1888.
109. **Liouville, R.**, Sur les lignes géodésiques des surfaces à courbure constante. American J., X., 1888.
110. **Voss, A.**, Ueber diejenigen Flächen, auf denen 3 Scharen geodätischer Linien ein conjugirtes System bilden. Munch Ber., 1888.

111. **Koenigs, G.**, Sur les surfaces, dont le ds^2 peut être ramené de plusieurs manières au type de Liouville. C. R., CIX., 1889.
112. **Raffy, L.**, Sur un problème de la théorie des surfaces. Darboux Bull., XIII., u. C. R., CVIII., 1889.
113. **Chini, M.**, Sulle superficie a curvatura media costante. Batt. G., XXVII., 1889.
114. **Guichard, C.**, Surfaces rapportées à leurs lignes asymptotiques. Ann. de l'Éc. Norm., VI., 1889.
115. **Reina, V.**, Sugli oricicli delle superficie pseudosferiche. Rom. Acc. L. Rend. (4), V_1., 1889.
116. **Razziboni, A.**, Delle superficie, sulle quale due serie di goedetiche formano un sistema coniugato. Bologna Mem. (4), IX., 1889.
117. ——— Sulla rappresentazione di una superficie su di un altra al modo di Gauss. Batt. G., XXVII., 1889.
118. **Bianchi, L.**, Ricerche sulle superficie elicoidali. Batt. G., XXVII., 1889.
119. **Guichard, C.**, Recherches sur les surfaces à courbure totale costante. Ann. de l'Éc. Norm. (3), VII., 1890.
120. ——— sur les surfaces, qui possèdent un réseau de géodésiques conjuguées. C. R., CX., 1890.
121. **Padova, E.**, Sopra un teorema di geometria differenziale. Ist. Lomb. Rend., XXIII., 1890.
122. **Bianchi, L.**, Sopra una classe di rappresentazioni equivalenti della sfera sul piano. Rom. Acc. L. Rend. (4), VI_1., 1890.
123. ——— Sopra una nuova classe di superficie appartenenti a sistemi tripli ortogonali. Rom. Acc. L. Rend. (4), VI_1., 1890.
124. ——— Sulle superficie, le cui linee assintotiche in un sistema sono curve a torsione costante. Rom. Acc. L. Rend. (4), VI_1., 1890.
125. ——— Sopra alcune nuove classi di superficie e di sistemi tripli ortogonali. Annali di Mat. (2), XVIII., 1890.
126. **Schwarz, H. A.**, Gesammelte Abhandlungen. II. Bd., S. 363 ff. 1890.
127. **Picard, E.**, Théorie des équations aux dérivées partielles. Journ. de Math. (4), VI., 1890.
128. **Ribancour, A.**, Mémoire sur la théorie générale des surfaces courbes. Journ. de Math. (4), VII., 1891.
129. **Voss, A.**, Zur theorie der Krümmung der Flächen. Math. Ann., XXXIX., 1891.
130. **Cosserat, E.**, Sur les systèmes cycliques et la déformation des surfaces. C. R., CXIII., 1891.
131. ——— Sur les systèmes conjugués et sur la déformation des surfaces. C. R , CXIII., 1891.
132. **Bianchi, L.**, Sui sistemi tripli ortogonali, che contengono una serie di superficie con un sistema di linee di curvatura plane. Annali di Mat. (2), XIX., 1891.
133. **Wangerin, A.**, Ueber die Abwickelung von Rotationsflächen mit constantem negativem Krümmungmass auf einander. Naturf. Ges. Halle, 1891.
134. **Hollaender, E.**, Ueber äquivalente Abbildung. Diss. Halle u. Progr. Mühlheim a. d. Ruhr, 1891.
135. **Padova, E.**, Di alcune classi di superficie suscettibili di deformazioni infinitesime speciali. Ist. Lomb. Rend. (2), XXIV., 1891.
136. **Backlund, A. V.**, Anwendung von Sätzen über partielle Differentialgleichungen auf die Theorie der Orthogonalsysteme. Math. Ann., XL., 1892.

137. **Bianchi, L.**, Sulla trasformazione di Bäcklund per le superficie pseudosferiche. Rom. Acc. L. Rend. (5), I_2., 1892.
138. ——— Sulla trasformazione di Bäcklund pei sistemi tripli ortogonali pseudosferici. Rom. Acc. L. Rend. (5) I_2., 1892.
139. ——— Sulle deformazioni infinitesime delle superficie flessibili ed inestendibili. Rom. Acc. L. Rend. (5), I_2., 1892.
140. **Koenings, G.**, Résumé d'un mémoire sur les lignes géodésiques. Toulouse Annales, VI., 1892.
141. **Cosserat, E.**, Sur la déformation infinitésimale et sur les surfaces associées de Bianchi. C. R., CXV., 1893.
142. ——— Sur les congruences des droites et sur la théorie des surfaces. Toulouse Annales, VII., 1893.
143. **Guichard, C.**, Sur les surfaces, dont les plans principaux sont équi distants d'un point fixe. C. R., CXVI., 1893.
144. **Waelsch, E.**, Sur les surfaces à élément linéaire de Liouville et les surfaces à courbure constante. C. R., CXVI., 1893.
145. ——— Ueber die Flächen constanter Krümmung. Wien Ber., CII., 1893.
146. **Probst, F.**, Ueber Flächen mit isogonalen Systemen von geod¨tischen Kreisen. Diss. Würzburg, 1893.
147. **Bianchi, L.**, Sulla interpretazione geometrica del teorema di Moutard. Rom. Acc. L. Rend. (5), III_2., 1894.
148. ——— Applicazioni geometriche del metodo delle approssimazioni successive di Picard. Rom. Acc. L. Rend. (5), III_2., 1894.
149. ——— Sulle superficie, i cui piani principali hanno costante il rapporto delle distanze da un punto fisso. Rom. Acc. L. Rend. (5), III_2., 1894.
150. ——— Sui sistemi tripli ortogonali di Weingarten. Palermo Rend., VIII., 1894.
151. **Soler, E.**, Sopra una certa deformata della sfera. Palermo Rend., VIII., 1894.
152. **Cosserat, E.**, Sur des congruences rectilignes et sur le problème de Ribaucour. C. R., CXVIII., 1894.
153. **Genty, E.**, Sur les surfaces à courbure totale constante. S. M. F. Bull., XXII., 1894.
154. **Wangerin, A.**, Ueber die Abwickelung von Flächen constanten Krümmungsmasses sowie einiger anderer Flächen auf einander. Festschift d. Univ. Halle 1894.
155. **Voss, A.**, Ueber isometrische Flächen. Math. Ann., XLVI., 1895.
156. **Filbi, C.**, Sulle superficie che, da un doppio sistema di traiettorie isogonali sotto un angelo costante delle linee di curvatura, sono divise in parallelogrammi infinitesimi equivalenti. Rom. Acc. L. Rend., IV_1., 1895.
157. **Busse, F.**, Ueber diejenige punktweise eindeutige Beziehung zweier Flächenstücke auf einander, bei welcher jeder geodätischen Linie des einen eine Linie constanter geodätischer Krümmung des anderan entspricht. Berl.-Ber.
158. ——— Ueber eine specielle conforme Abbildung der Flächen constanten Krümmungsmasses auf die Ebene. 1896.
159. **Darboux, G.**, Leçons sur la théorie générale des surfaces. Bd. I.–IV., 1887–1896.
160. **Bianchi, L.**, Lezioni di geometria differenziale. Gött., 1894.
161. **Genty, E.**, Sur la déformation infinitésimale des surfaces. Toulouse Ann., IX., 1896.
162. **Goursat, E.**, Sur les lignes asymptotiques. C. R., CXXII., 1896.

163. **Calinon, A.**, Le theorème de Gauss sur la courbure. Nouv. Ann., 13, 1895.
164. **Weingarten, J.**, Sur la déformation des surfaces. Jour. de Math. (5), 1896.
165. **Bianchi**, Nuove ricerche sulla superficie pseudo sferiche. Annali di Mat., 24, 1896.
166. ——— Sopra una classe die superficie collegate alle superficie pseudo sferiche. Rom. Accad. Lin. (5), 5, 1896.
167. **Klein, F.**, Lie's Transformation. Math. Ann., 50, 1897.
168. **P. Stackel und Fr. Engel**, Gauss, die beiden Bolyai und die Nicht-Euklidische Geometrie. Math. Ann., 49, 1897.
169. **Bukreiew**, Flächenelement der Fläche constanter Krümmung. Kiew Univ. Nachr, No. 7, 1897.
170. **Voss, A.**, Ueber infinitesimalen Flächen Deformationen. Deutsche Math., Vol. 4, 1896.
171. ——— Zur Theorie der infinitesimalen Biegungdeformationen einer Fläche. Münch Ber., 27, 1897.
172. **Bianchi, L.**, Sur deux classes de surfaces qui engendrent par un movement helicoidal une famille de Lamé. Toulous Ann., 11, 1898.
173. **Darboux, O.**, Leçons sur les systèmes orthogonaux et les coördinées curvilignes. Paris, 1898.
174. **Guichard, C.**, Sur les surfaces à courbure totale constante. C. R., 126.
175. ——— Sur les systèmes orthogonaux et les systèmes cycliques. Ann. de l'Ec. Nom. (3), 15, 1898.
176. **Carda, K.**, Zur Geometrie auf Flächen constanter Krümmung. Wein Ber., 107, 1898.
177. **Metzler, G. F.**, Surfaces of rotation with constant measure of curvature. Am. J. of Math., 20, 1898.
178. **Tzitzeia**, Sur les surfaces à courbure totale constante. C. R., 128, 1898.
179. **Darboux, G.**, Sur la transformation de M. Lie et les surfaces enveloppées de sphères. Darboux Bull. (2), 1897.
180. **Kommerell, V.**, Bemerkung zur den asymptotenlinien. Böklen Math. (2), 2, 1900.
181. **Waelsch, E.**, Ueber Flächen mit sphärischen oder ebenen Krümmungslinien. Hoheschule Brünn, 133, 1899.
182. **Hilbert, D.**, Ueber Flächen von constanter Gaussicher Krümmung. Trans. Am. Math. Soc. (2), 1, 1901.
183. **Zuhlke, P.**, Ueber die geodätischen linien und Dreiecke auf den Flächen Constanten Krümmungsmasses und ihre Beziehung zur die sogenannten nicht-Euclidischen geometrie. Charlottenburg, 1902.

NOTE. The above list of books is based upon the one given by Busse at the end of his essay entitled "*Ueber eine specielle conforme Abbildung der Flächen constanten Krümmungsmasses auf die Ebene.*"[157]

In the following pages the notation of the various authors quoted has been translated into the notation used by Bianchi in his book entitled, *Lezioni di geometria differenziale*,[160] since the difference in the notation used by the different writers is not of special interest.

A list of some of Bianchi's formulæ is added at the end of this paper.

I.

THE APPLICATION OF ONE PSEUDOSPHERICAL SURFACE UPON ANOTHER AND THE GEOMETRY OF THE SURFACES.

1. Surfaces whose measure of curvature at every point is constant and negative were called pseudospherical surfaces by Beltrami in 1868, in order, as he said, "to avoid circumlocution." Since, therefore, the definition of these surfaces depends upon the definition of the measure of curvature itself, their history may be considered as commencing in 1827, when Gauss[2] in his great memoir entitled "Disquisitiones generales circa superficies curvas" established the idea of curvature as it is understood today.

2. In this famous paper Gauss borrowed from the astronomers the notion of spherical representation and established a point-to-point correspondence between a curved surface and a sphere of unit radius. He supposed a radius of the sphere to be drawn parallel to the assumed positive direction of the normal to the curved surface at a point P, and the extremity of the radius to be a point p corresponding to P of the surface. He defined the total curvature of a part of the surface enclosed within certain limits as the area of the figure on the sphere corresponding to it, and distinguished this curvature from the very important notion of the measure of curvature of the surface at a point, which is sometimes also called total curvature. This last is defined as the quotient of the total curvature of the surface element at the point by the area of the surface element, or in other words, "the ratio of the infinitely small areas that correspond to one another on the curved surface and on the sphere." He remarked further that "the position of a figure on the sphere can be either similar to the position of the corresponding figure on the curved surface or the inverse." When the position of two corresponding figures, the one on the surface, the other on the sphere, is similar, he called the curved surface a convexo-convex surface, or a surface with positive curvature. When the position of the figure is inverse to that of the figure on the surface he called the surface a concavo-convex surface or a surface with negative curvature.

Gauss introduced various analytic expressions for this measure of curvature at a point which he denoted by K, among others, using a general parametric representation of a surface through the parameters u, v, he found that

$$K = \frac{DD'' - D'^2}{EG - F^2},$$

where D, D', D'', E, F and G, functions of u and v, are the coefficients of

the first and second fundamental differential expressions of the surface, the one for the square of the linear element

$$ds^2 = Edu^2 + 2Fdudv + Gdv^2,*$$

the other

$$-\Sigma dx dX = Ddu^2 + 2D'dudv + D''dv^2,$$

where X, Y, Z are the direction cosines of a normal to the surface at a point x, y, z.

When he chose as a special system of parametric lines a family of geodesic lines and their orthogonal trajectories, he showed that E becomes a function of u alone and F vanishes, so that the expression for the line element assumes the form

$$ds^2 = du^2 + Gdv^2,$$

and he could derive for the curvature a simple corresponding form

$$K = -\frac{1}{\sqrt{G}} \frac{\partial^2 \sqrt{G}}{\partial u^2}.$$

In particular he observed that if the system is a geodesic polar system in which the v-curves proceed from a point and v is the angle that each geodesic v-curve makes with an arbitrary but fixed v-curve and u is the arc-length of each geodesic from the point, then G is a function which satisfies the equations

$$(\sqrt{G})_{u=0} = 0, \quad \left(\frac{\partial \sqrt{G}}{\partial u}\right)_{u=0} = 1,$$

and that if the v-curves form a geodesic parallel system, that is, if the v geodesics are orthogonal to a geodesic curve $u = 0$ and u is as before the arc-length along the v curves from $u = 0$ and the arc-length v is measured on the curve $u = 0$ from some fixed point, the function G satisfies the equations

$$(\sqrt{G})_{u=0} = 1, \quad \left(\frac{\partial \sqrt{G}}{\partial u}\right)_{u=0} = 0.$$

Lastly he wrote

$$K = \frac{rt - s^2}{(1 + p^2 + q^2)^2},$$

where the surface is represented by

$$z = f(x, y),$$

and p, q, r, s, t have their usual meaning as the partial derivatives of z with respect to x and y.

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* $$E = \Sigma\left(\frac{\partial x}{\partial u}\right)^2, \quad F = \Sigma\frac{\partial x}{\partial u}\cdot\frac{\partial x}{\partial v}, \quad G = \Sigma\left(\frac{\partial x}{\partial v}\right)^2,$$

$$D = -\Sigma\frac{\partial X}{\partial u}\cdot\frac{\partial x}{\partial u}, \quad D' = -\Sigma\frac{\partial X}{\partial v}\cdot\frac{\partial x}{\partial u} = -\Sigma\frac{\partial X}{\partial u}\cdot\frac{\partial x}{\partial v}, \quad D'' = -\Sigma\frac{\partial X}{\partial v}\cdot\frac{\partial x}{\partial v},$$

Bianchi[160], §§ 33, 46.

The measure of curvature at a point as defined by Gauss has now been adopted as the standard definition and called the Gaussian measure of curvature or total curvature. Both before and after the time of Gauss various definitions of curvature of a surface had been advanced by Euler, Meusnier, Monge and Dupin, but these definitions have not recommended themselves and are now almost forgotten.

Gauss did not write directly on the subject of pseudospherical surfaces, but in his memoir just quoted he published two important discoveries which were afterwards easily applied to the special case of these surfaces. To Gauss is due the celebrated theorem on the total curvature, (curvatura integra), of a geodesic triangle, for making use of the geodesic polar system he found that the total curvature of a triangle whose angles are A, B, C is

$$A + B + C - \pi$$

which is negative for surfaces of negative curvature and positive for surfaces of positive curvature. An immediate inference from this and what may be regarded as the first theorem in the geometry of pseudospherical surfaces is that the area of a geodesic triangle on one of these surfaces is proportional to its spherical deficiency. This theorem was proved by Bertrami in 1868.*

3. Gauss also established the well-known theorem that K is an invariant of bending, that is, any disturbing of the shape of the surface which does not involve stretching or crushing, leaves the value of K at any point unaltered. Thus if one surface is applicable upon another the measure of curvature at corresponding points of the two surfaces is the same. It was this invariant character of K that first gave interest to the study of surfaces of constant curvature. Gauss himself made no study of them, but Minding[6] in a paper of 1839, of which more will shortly be said, discussed the sufficiency of Gauss' theorem for the applicability of one surface upon another, and established that for surfaces of constant curvature, and for these only, is Gauss' theorem a sufficient as well as a necessary condition.

In a previous paper in 1830 he had integrated the Gaussian equation

$$K = -\frac{1}{\sqrt{G}} \frac{\partial^2 \sqrt{G}}{\partial u^2},$$

assuming K to be constant, and had obtained the expression for the linear element

$$ds^2 = du^2 + \frac{[\sin(u\sqrt{K})]^2}{K} dv^2.$$

When about to apply one surface upon another he accordingly wrote for the expressions for their linear elements

* Page 37.

$$ds^2 = du^2 + \left(\frac{\sin(u\sqrt{K})}{\sqrt{K}}\right)^2 dv^2,$$

$$ds'^2 = du'^2 + \left(\frac{\sin(u'\sqrt{K})}{\sqrt{K}}\right)^2 dv'^2,$$

in which the primes indicate the elements with reference to the second surface. The analytical condition of applicability

$$ds = ds'$$

be satisfied by putting

$$u = u', \qquad v = a + v'$$

where a may have any value from zero to infinity. The first equation shows that any point on the first surface may be made to correspond with any point on the second surface and the second equation that any geodesic curve on the first surface proceeding from the point may be made to correspond with any geodesic curve on the second surface proceeding from a corresponding point. Thus the surfaces are applicable upon each other in ∞^3 ways, or to quote Minding, "One can place two arbitrary points of the one upon two arbitrary points of the other, provided that the lengths of the shortest lines upon the surface between the pairs of points are equal to each other."

Becoming interested in the study of surfaces of constant curvature Minding[6] proceeded to determine some of these surfaces. When the surface is assumed to be of the form of $z = f(x, y)$, the differential equation of his problem is

$$\frac{\dfrac{\partial^2 z}{\partial x^2}\dfrac{\partial^2 z}{\partial y^2} - \left(\dfrac{\partial^2 z}{\partial x \partial y}\right)^2}{\left[1 + \left(\dfrac{\partial z}{\partial x}\right)^2 + \left(\dfrac{\partial z}{\partial y}\right)^2\right]^2} = K,$$

where $K =$ a constant.

"This integration," he said, "has never been effected up to the present time except for $K = 0$." Changing the form of the equation by writing $x = r\cos\psi$ and $y = r\sin\psi$ in the place of x and y, he attempted its solution only for the special case

$$\frac{\partial z}{\partial \psi} = h \qquad (h = \text{a constant}).$$

A first integration gave him

$$dz = h\,d\psi \pm \left(\sqrt{\frac{1}{a^2 + r^2} - 1 - \frac{h^2}{r^2}}\right) dr \quad \left(\begin{matrix} K = -1 \\ a^2 = \text{constant of integration}\end{matrix}\right).$$

For surfaces of constant negative curvature he first put $a^2 = 0$ in this equation of z and retained h: then he let h vanish and a^2 remain. When the first condition is fulfilled, he said, "the equation for z represents a curve which generates

the surface in the same way as the straight line generates the helicoidal surface namely, by a revolution about the axis of z while at the same time all its points have a common motion parallel with z." This surface was afterwards investigated by Dini [24, 25] and called the Dini helicoidal surface. When the second condition exists, Minding found that the surface becomes a surface of rotation and the z curves become its meridian lines. He discovered three types of these surfaces, those for which a^2 has a positive value, those for which it has a negative value and those for which it is equal to zero. It was probably his original expression for the linear element

$$ds^2 = du^2 + \left(\frac{\sin(u\sqrt{K})}{\sqrt{K}}\right)^2 dv^2$$

or

$$ds^2 = du^2 + \sinh^2 u dv^2 \qquad (K = -1)$$

that suggested to him the expedient of putting $r = \sinh\phi$ in the equation for z when a^2 is positive, so that

$$z = \int \pm (\sqrt{1 - a^2 \sinh^2 \phi})\, d\phi,$$

while for a negative value for a^2 he naturally introduced the analogous expression

$$z = \int \pm (\sqrt{1 - a^2 \cosh^2 \phi})\, d\phi,$$

Finally, when $a^2 = 0$, he put

$$r = \frac{1}{\cosh\phi}, \qquad z = \phi - \tanh\phi.$$

This last equation for z shows that it represents a tractrix but Minding did not call the curve by its name, although he remarked "that it approaches the axis of rotation asymptotically." He drew pictures of the meridian curves of these three types and the plates appear at the end of his article in Crelle's Journal. In this way he obtained two classes of pseudospherical surfaces, — the helicoidal surfaces and the three types of surfaces of rotation.

Knowing that all the three forms of surfaces of rotation of curvature -1 corresponding to the three types of the meridian curves are deformable into each other, he[7] wished to bring their linear element into the same form

$$ds^2 = du^2 + \sinh^2 u dv^2.$$

He perceived that this could be easily accomplished for surfaces of the first kind but that for those of the second or third kind it is necessary to find for the parametric lines a system of geodesic lines that go out from the same point. He developed a general expression for the linear element of a pseudospherical surface

$$ds^2 = (v^2 + h^2 - a^2)\,dt^2 + \frac{1}{v^2 + h^2 - a^2}\,dv^2,$$

by writing first

$$ds^2 = dx^2 + dy^2 + dz^2 = dr^2 + r^2 d\psi^2 + \left[h\,d\psi + \left(\sqrt{\frac{1}{r^2 + a^2} - \frac{h^2}{r^2} - 1}\right) dr\right]^2,$$

and then putting

$$dt = d\psi + h \left[\frac{\dfrac{1}{r^2 + a^2} - \dfrac{h^2}{a^2} - 1}{h^2 + r^2}\right] dr.$$

and

$$v = r^2 + a^2.$$

The next year he worked out a set of equations for transforming this general expression for the linear element into the one referred to a geodesic polar system

$$ds^2 = d\sigma^2 + \sinh^2 \sigma d\theta^2.$$

Taking two points corresponding to coördinates (t, v), (t', v') of the same system and denoting the length of the geodesic line which connects them by σ and the angle which this line makes with the curve $v =$ a constant by θ, he stated that the variables v, t, σ, θ are related by the equations

$$\frac{bv' \operatorname{ctn} \theta - b^2 \tan b\,(t - t')}{b + v' \operatorname{ctn} \theta \cdot \tan b\,(t - t')} = \frac{\cos\theta\,(\sqrt{v'^2 - b^2}) \cdot [v' + \tanh\sigma \cdot \sin\theta \cdot \sqrt{(v'^2 - b^2)}]}{\sin\theta(\sqrt{v'^2 - b^2}) + v' \tanh\sigma}$$

and

$$v = \sinh\sigma \sin\theta \cdot \sqrt{v'^2 - b^2} + v' \cosh\sigma$$

where

$$b^2 = a^2 - h^2 \gtreqless 0 \text{ and } h = 0$$

and that these become

$$v \cdot (t - t') = \sinh\sigma \cos\theta,$$

$$v = v'(\cosh\sigma + \sinh\sigma \sin\theta)$$

when $b = 0$.

He did not formally state the conclusion of this argument that the two sets of coördinates v, t and σ, θ may be regarded as lying on two different surfaces and that then these equations, instead of changing the expression for the linear element of one surface only into a slightly different form, will actually transform one surface into another, and that, if the first surface is supposed to be a surface of rotation for which $t =$ a constant and $v =$ a constant are the meridian curves and the parallels respectively, the surface will be one of the three types of which he spoke in his earlier paper according as to whether

$$a^2 \gtreqless 0,$$

so that the above equations will then transform a surface of one of those three types into one of the first type.

Eleven years after the publication of Minding's paper on the applicability of surfaces, Liouville [10] in the fourth note to his edition of Monge's work on the "Application of Analysis to Geometry" sought to solve the same problem, whether the fact that two surfaces have the same measure of curvature at corresponding points is a sufficient as well as a necessary condition for the deformability of one into the other. His investigations led him to criteria of applicability equivalent to Minding's. No reference is made to Minding's paper, but the methods used differ from that writer's. As Liouville had employed in the general discussion curves of constant measure of curvature as a part of his coördinate system the results could not be extended immediately to surfaces of constant curvature, he therefore made a separate study of these surfaces. No new theorems were developed, but the work has special interest in that he employed isothermal and minimal curves and made use of a new form for the expression for curvature. Taking the linear element in the form

$$ds^2 = \lambda(d\alpha^2 + d\beta^2) \qquad \text{or} \qquad ds^2 = \lambda(du\,dv)$$

and writing

$$u = \alpha + i\beta, \qquad v = \alpha - i\beta$$

he obtained a corresponding expression for total curvature

$$\frac{\partial^2 \log \lambda}{\partial u \partial v} - \frac{\lambda}{2a^2} = 0,$$

which is a differential equation for the determination of λ for K constant and equal to $-1/a^2$, and whose resulting integral * is

$$\lambda = \frac{4a^2 e^{2\zeta}\left(\frac{\partial(\zeta + i\tau)}{\partial u}\right)\left(\frac{\partial(\zeta - i\tau)}{\partial v}\right)}{(1 \pm e^{2\zeta})^2}$$

where 2ζ is the sum of a function of u and a function of v, and 2τ is the difference.

His corresponding expression for the square of the linear element is

$$ds^2 = \frac{4a^2 e^{2\zeta}}{(1 \pm e^{2\zeta})^2}(d\zeta^2 + d\tau^2),$$

which he reduced to the still simpler form,

$$ds^2 = \frac{a^2 dr^2}{a^2 b^2 \mp r^2} + r^2 d\theta^2,$$

* N. B. LIOUVILLE first considered the question of the converse of the theorem of GAUSS in a paper published in the Journal de Mathématiques, Paris, vol. 12. The question was taken up and further developed by BERTRAND, PUISSEUX and DIGUET respectively in papers published in vol. 13 of the same Journal. LIOUVILLE quoted these writers in the new edition of his paper which appeared as Note IV in the Appendix to MONGE'S *Application de l'Analysis à la Géometrie*, and in this note first developed the formula for the pseudosphere. In a third paper, very brief in length, published in the Journal de Mathématiques, vol 18, he gave the complete details of the integration of the equation.

by putting $b\theta$ for τ and r for $\frac{2ab\epsilon^{\zeta}}{1 \pm \epsilon^{2\zeta}}$.

Finally, writing

$$w + \theta i = K\frac{(1 - e^{\zeta + i\tau})}{(1 + e^{\zeta + i\tau})},$$

and decomposing the equation into two real equations, he reduced it to

$$ds^2 = \frac{a^2}{w^2}(dw^2 + d\tau^2).$$

The same expression results when, in addition to the constant K, constants g and h are introduced in putting

$$w + (\theta + g)i = K\frac{1 - e^{\zeta - (\tau + h)i}}{1 + e^{\zeta - (\tau + h)i}}.$$

In this manner he established Minding's theorem that surfaces of the same curvature are applicable upon one another in ∞^3 of ways. The simplest surface of constant negative curvature upon which all the surfaces of the same curvature are applicable he found to be the surface generated by the tractrix revolving about its asymptote.

Codazzi [15] was the next mathematician after Liouville to make a study of pseudospherical surfaces. He was the first to derive the expression for the linear element of the surface referred to a geodesic parallel system in the form

$$ds^2 = du^2 + \cosh^2 u dv^2.$$

He integrated Gauss' expression for curvature

$$K = -\frac{1}{\sqrt{G}}\frac{\partial^2 \sqrt{G}}{\partial u^2}$$

with the conditions that

$$[\sqrt{G}]_{u=0} = 1, \qquad \left[\frac{\partial \sqrt{G}}{\partial u}\right]_{u=0} = -\frac{1}{\rho}$$

where ρ is the radius of geodesic curvature of $u = 0$, and obtained

$$\sqrt{G} = \cosh u - \frac{1}{\rho}\sinh u.$$

When $u = 0$ is a curve of constant geodesic curvature, $1/\rho = 0$, a hypothesis that he makes in connection with surfaces of revolution and the formula for ds^2 follows.

His chief contribution to the subject was to the pseudospherical trigonometry. He developed Minding's formula for showing the relation between the sides and angles of a geodesic triangle

$$\cosh a = \cosh b \cosh c - \sinh b \sinh c \cos A,$$

and also the well known formula

$$\frac{\sinh a}{\sin A} = \frac{\sinh b}{\sin B} = \frac{\sinh c}{\sin C},$$

both of which are analogous to those of spherical trigonometry.

Choosing for his coördinates a geodesic polar system and writing the expression for the linear element in the form

$$ds^2 = du^2 + \sinh^2 u dv^2,$$

he took for his triangle the area bounded by two geodesic lines going out from the origin, $v = 0$ and $v =$ a constant, and by a third geodesic line making an angle $\pi - \beta$ with $v = 0$ at a distance α from the origin and an angle θ with $v =$ a constant. He then integrated Gauss' equation for a geodesic line which makes an angle θ with any parametric geodesic line $v =$ a constant,

$$d\theta + \frac{\partial \sqrt{G}}{\partial u} dv = 0.$$

By means of his equation for the linear element and the equation for $\cos\theta$,

$$\cos\theta = \frac{du}{ds},$$

he found that Gauss' equation becomes

$$ctnh\, u\, du + ctn\, \theta d\theta = 0$$

so that $\sinh u \sin\theta = a$ constant $= \sinh\alpha \sin\beta$. He was then able to transform the equation for $d\theta$ into

$$ds = \frac{\sinh u du}{\sqrt{\cosh^2 u - (\sinh^2\alpha \sin^2\beta + 1)}}$$

which, when integrated, gives

$$\cosh u = \cosh\alpha \cosh s - \sinh\alpha \sinh s \cos\beta,$$

where u, α and s are the arc-lengths that form the sides of the triangle and β is the angle opposite the side u.

4. Dini[22, 24, 25, 29] was acquainted with the work of Minding and in the years 1865 and 1866 he wrote four papers in which among other things he developed more fully the subjects touched upon by the earlier mathematicians, the equation transforming a pseudospherical surface of rotation of one of the three types into that of another and the determination of the form of a helicoidal surface with constant negative curvature.

His method of finding the linear element[24] of a surface of rotation with constant curvature as it is set forth in his second paper is perhaps simpler than

that of his predecessors. Taking the differential equation for the length of the radius of curvature of a meridian curve at a certain point, and the one for the length of the normal between the same point on the curve and the axis of the revolution, he found the equation for the total curvature $-1/a^2$ to be

$$\frac{\dfrac{dz}{dr} \cdot \dfrac{d^2z}{dr^2}}{\left[1 + \left(\dfrac{dz}{dr}\right)^2\right]^2 r} = -\frac{1}{a^2},$$

which, when integrated, becomes

$$\frac{dz}{dr} = \sqrt{\frac{a^2}{r^2 + l} - 1},$$

where r is the radius of the parallel circle, z is the axis of revolution, $z = \phi(r)$ is the equation of the meridian curve and l is a constant of integration. He gave to l the three values successively

$$\alpha^2, \qquad 0, \qquad -\alpha^2,$$

and thus found three forms for the meridian curve,

(1) $$\frac{dz}{dr} = \sqrt{\frac{a^2}{r^2 + \alpha^2} = 1},$$

(2) $$\frac{dz}{dr} = \sqrt{\frac{a^2}{r^2} - 1},$$

(3) $$\frac{dz}{dr} = \sqrt{\frac{a^2}{r^2 - \alpha^2} - 1},$$

and three corresponding forms for the linear element

(1) $$ds^2 = du^2 + \alpha^2 \sinh^2 \frac{u}{a}\, dv^2,$$

(2) $$ds^2 = du^2 + \beta^2 \epsilon^{2u/a}\, dv^2,$$

(3) $$ds^2 = du^2 + \alpha^2 \cosh^2 \frac{u}{a}\, av^2,$$

where u and v represent the parallel and meridian curves respectively. "The surfaces (1), (2) and (3)," he said, "are the only surfaces of revolution with constant negative curvature. They may be applied one on the other, but as we shall see they constitute as to their application three classes of sharply distinct surfaces. The second of these classes ($l = 0$) is made up of a single surface, the surface which has for its meridian the curve with tangents of constant length,

as already considered by Liouville, and it serves as a transition from surfaces of the first class to those of the third."

In the special case when $\alpha^2 = a^2$ he called the first surface "the imaginary sphere," a surface which was afterwards studied by Beltrami[48] and Cayley[87].*

Dini next took up the problem of the application of these surfaces upon one another. He remarked that if in applying one surface upon a second the meridian curves of the two coincide, both surfaces must be of the same type. He then proceeded to examine the nature of the curves on the deformed surface into which the meridian curves of the original surface pass when the two surfaces are of different types. For this purpose he first sought the equations which transform a surface of the first or third type into a pseudosphere.

When a surface of the first type is applied upon one of the second, he put r_1/α for $\sinh(u/a)$, for by means of this substitution the linear element of the first surface becomes

$$ds^2 = \frac{a^2\,dr_1^2}{r_1^2 + \alpha^2} + r_1^2\,dv^2.$$

a form given by Liouville for the pseudosphere, and the meridian curves and the parallel circles go over into a family of geodesics meeting at a point and their orthogonal trajectories. He found that the deformation of a surface of the third type into a pseudosphere is more complicated. In his first paper[22, p. 66] he had developed the important theorem that a surface that possesses a system of geodesics intersected at equal distances by curves each of which meets the geodesics at the same constant angle, not a right angle, and varying by a fixed law for each curve, is either a surface of rotation or applicable upon one. He transformed a surface with constant curvature referred to such a system of geodesics and their trajectories for parameters into a surface of rotation by two methods. By the application of the first method the trajectories of the geodesics on the first surface pass over into parallel circles on the surface of rotation and the family of curves that cut the trajectories orthogonally become its meridian curves. He found not only the equations for this transformation but also those for the reverse operation, [22, p. 68] for deforming a surface of rotation in such a way that its parallels go over into a system of trajectories intersecting equal lengths on a family of geodesic lines.

On the other hand, when he deformed the geodesic lines on his original surface into the meridians of a surface of rotation, the trajectories became loxodromes. He found that this last transformation leads to but one surface of rotation, the pseudosphere, and he introduced in this connection for its linear element the now well known expression [22, p. 70]

$$ds^2 = du^2 + e^{2u}dv^2,$$

which did not appear in any of the papers previously discussed.

* P. 40.

It was probably this last result that suggested to him [24, p. 244] transforming a pseudospherical surface of rotation of the third type whose linear element is

$$ds^2 = du^2 + \alpha^2 \cosh^2 \frac{u}{a}\, dv^2$$

into a pseudosphere whose linear element is

$$ds^2 = du_1^2 + \beta^2 e^{2u_1/a} dv_1^2$$

by means of the equations

$$v = \int \frac{du}{\alpha \cosh \frac{u}{a} \sinh \frac{u}{a}} + \frac{t}{\alpha}$$

$$\log \sinh \frac{u}{a} = \frac{u_1 + v_1}{a}$$

$$t = -v_1.$$

He found that the parallels of the first surface pass over into loxodromes on the pseudosphere and that the meridians of the first surface pass into the orthogonal trajectories of these loxodromes. He proved that the cosine of the angle at which these loxodromes meet the meridian curves is equal to cosh u/a, that they possess a constant geodesic curvature and that their orthogonal trajectories are geodesic lines.

He remarked that the intermediate surface, obtained by using the first equation of transformation only, is the screw surface of constant negative curvature whose helices correspond to the loxodromes of the pseudosphere.

Dini[29] was very much interested in surfaces of this nature, to which he devoted one paper exclusively in 1866, but before taking up the examination of its contents a digression will here be made for the consideration of contributions of Bianchi and Beltrami to Dini's other theorems. Bianchi [63] introduced a general method for transforming a surface of rotation of any of the three types into a pseudosphere.

Instead of choosing a new system of geodesics on the original surface and then actually bending it until these curves become the meridians of the new surface, Bianchi followed Liouville's suggestion and used minimal lines for parameters, so that, when the surface is bent, the lines of reference remain the same.

Referring both surfaces to minimal lines he denoted the linear element of the surface which he wished to deform by

$$ds^2 = \frac{4A^2}{(x+y)^2}\, dxdy,^*$$

and changed the expression for the linear element of the pseudosphere,

* P. 15.

$$ds^2 = du_1^2 + e^{\frac{2u_1}{A}} dv_1^2$$

into

$$ds^2 = \frac{4A^2}{(\alpha+\beta)^2} d\alpha d\beta,$$

where $-1/A^2$ is the total curvature of both surfaces.

His equations of transformation were then

$$\alpha = \frac{1}{ax+b} + c, \qquad \beta = \frac{1}{ay-b} - c \quad (a,\ b \text{ and } c \text{ are constants}),$$

and by writing for α and β their values in terms of u and v and for x and y their values in terms of u and v, the parallel and meridian curves of the surface to be deformed, he found three sets of equations for transforming a surface of the first, second and third type respectively into a pseudosphere:

(1)
$$v_1 e^{\frac{u_1}{a}} = a \sinh \frac{u}{a} \sin v,$$
$$e^{\frac{u_1}{a}} = \frac{a}{K}\left(\sinh \frac{u}{a} \cos v + \cosh \frac{u}{a}\right).$$

(2)
$$e^{\frac{u_1}{a}} = \frac{e^{-u/a}}{v^2 + a^2 e^{-2u/a}}.$$
$$v_1 = \frac{v}{v^2 + a^2 e^{-2u/a}}.$$

(3)
$$u_1 = a \log \cosh \frac{u}{a} + av$$
$$v_1 = a \tanh \frac{u}{a} - e^{\frac{v}{a}}$$

The first of these sets of equations is the same as the one given by Minding,* if in the latter the point (1, 0) be chosen for the point (v, t). The third set is identical with Dini's, for expressed in the same notation Dini's equations become

$$v = \log \tanh u - \log v_1, \qquad u_1 + \log v_1 = \log \sinh u,$$

and when one is subtracted from the other,

$$u_1 = \log \cosh u + v.$$

The validity of transforming the general expression for the linear element

$$ds^2 = Edu^2 + 2Fdudv + Gdv^2$$

* Page 14.

into the form in terms of the conjugate complex variables θ and θ_0

$$ds^2 = \frac{d\theta \cdot d\theta_0}{\left(\frac{\kappa}{4} + \theta\theta_0\right)^2} \qquad (\kappa = \text{curvature})$$

was investigated by Weingarten[79] three years later. He decided that such a transformation is permissible for surfaces of constant curvature, and that then the reciprocal of the differential quotient $\partial\theta/\partial u$ must satisfy two partial differential equations.

The geometrical interpretation of the difference in the value of the constant in the equation for the meridian curve of the three types of surfaces of rotation was clearly determined by Beltrami.[32] The early papers of this mathematician on the subject of pseudospherical surfaces were contemporaneous with those of Dini and both appeared side by side in the Italian journals during the years 1864 and 1865.

Beltrami[26] wrote a long treatise on the general theory of surfaces in which he considered in particular geodesic curvature, evolute and involute surfaces and differential parameters. He applied the various theorems that he obtained to the special case of pseudospherical surfaces, all of which theorems will be discussed in detail in the chapter on Evolute Surfaces and the Transformation Theory. In the same year that this treatise appeared Beltrami[21] published a paper devoted entirely to the pseudosphere in which he investigated its geometrical properties.

He gave a geometrical proof of Dini's statement that the second class of pseudospherical surfaces of rotation is composed of a single surface, which is equivalent to saying that the pseudosphere is always bent into a pseudosphere, that is, it is identical with itself, if bent when the meridians are retained as meridians. He first showed that the radius of geodesic curvature of every parallel of a pseudosphere, being equal to the length of the tangent to the meridian curve between its point of tangency and the axis of revolution, is a constant R where $-1/R^2$ is the total curvature of the surface. Having proved that geodesic curvature is an invariant of bending, he then observed that, when the pseudosphere is so deformed that its meridian curves remain meridian curves, the radius of a geodesic curvature of every parallel circle of the deformed surface is R, that its meridians are therefore curves with tangents of constant length, and that they must therefore be identical in form with the meridians of the original surface, since to one value for the tangent length there corresponds but one tractrix.

He also showed, at this time, that the area and volume of a pseudosphere are equal to those of a sphere with the same numerical value of curvature, thus making an analogy between the simplest forms of surfaces of rotation with constant positive curvature and constant negative curvature.

Beltrami's [32] most important paper in regard to surfaces with constant negative curvature was published in 1868, under the title of an *Essay on the Interpretation of the Non-Euclidean Geometry.* Nothing written on the subject since Minding's [6] paper in 1839 can be compared in importance with this cele brated memoir by Beltrami, in which for the first time was made clear the relation of pseudospherical geometry to the general theory of geometry. The contents of Beltrami's paper will be given later; here attention only will be called to the fact that, although the greater part of the essay is devoted to the demonstration of geometrical propositions, yet it contains a proof of his discovery that the difference in the expression for the linear element of the surface of rotation results from the difference in the nature of the parallel circles chosen for one system of parameters, that the centers of those circles may be real finite points, points at infinity or imaginary points, and that consequently the corresponding surface will be of the first, second or third form given by Dini.

To return to the development of the helicoidal surface with constant negative curvature it will be remembered that Dini [25, 29] made a special study of this surface. He first communicated his results to the French Academy in 1865, and there stated that the surface is generated by a tractrix moving along a helix that lies on a cylinder. Recalling Bour's [19] Theorem, that helicoidal surfaces are applicable upon surfaces of rotation, he divided them into two classes, according to whether they are applicable upon a sphere or a pseudospherical surface. He used the ordinary expressions for a point on a helicoidal surface,

$$x = u \cos v, \quad y = u \sin v, \quad z = mv + \phi(u)$$

where m multiplied by 2π is the rise of the helix and $\phi(u)$ is a function that determines the form of the generating profile. He found that this generating profile $\phi(u)$ must satisfy the differential equation

$$u^2 + m^2 + u^2\left(\frac{d\phi}{du}\right)^2 = \frac{a^2K^2u^2}{K^2m^2 + K^2u^2 - 1} \qquad (K = \text{constant}),$$

in the case of surfaces of negative constant curvature $-1/a^2$. This he reduced to

$$u^2\left[1 + \left(\frac{d\phi}{du}\right)^2\right] = a^2 - m^2,$$

by putting $1/m^2$ for K^2.

Since the left hand member of this last equation is the square of the length of the tangent to the generating curve between the axis of revolution and its point of contact and the right hand member is a constant, Dini thus obtained an infinity of new helicoidal surfaces of negative curvature $-1/a^2$, each corresponding to a value of m and generated by a tractrix of tangent length $\sqrt{a^2 - m^2}$ moving about a cylinder and all developable upon the same pseudosphere.

5. Up to 1868 the only surfaces of constant negative curvature that had been determined and studied were surfaces of rotation and helicoidal surfaces. To these were added a new group of surfaces by Enneper[35, 42] in 1868. In a memoir written in that year he determined all the surfaces of constant curvature, one of whose families of lines of curvature is composed of plane curves or of spherical curves. As he showed, the presence of a system of plane lines of curvature on a surface of constant curvature requires that the second system of lines of curvature should be spherical, and conversely, moreover, the planes of the one system meet in a straight line and the centers of the spheres of the other systems lie on that line. The surfaces possessing these characteristics, whether of positive or negative constant curvature, have since been called Enneper's surfaces.

The determination of surfaces with either plane or spherical lines of curvature had already been attacked by Joachimsthal, Bonnet and Serret, and Bonnet[9] in particular examined surfaces for which the lines of curvature of the one family are plane and those of the other family are spherical, and showed that for surfaces of rotation "the lines of the one system are in planes all of which pass through the same straight line and the centres of the spheres on which are traced the spherical lines of curvature can lie on a right line."

Enneper considered surfaces of constant curvature not of rotation and set

$$\frac{1}{R_1 R_2} = -\frac{1}{g^2},$$

assuming u and v as the parameters of the lines of curvature of which $u =$ a constant are the plane lines of curvature and R_1 and R_2 are the principal radii of curvature. Putting

$$(1) \qquad \frac{1}{R_1} = -\frac{1}{g}\frac{1-t}{1+t}, \qquad \frac{1}{R_2} = \frac{1}{g}\frac{1+t}{1-t},$$

and employing the equations

$$\frac{\partial}{\partial v}\frac{\sqrt{E}}{R_1} = \frac{1}{R_2}\frac{\partial\sqrt{E}}{\partial v}, \qquad \frac{\partial}{\partial u}\frac{\sqrt{G}}{R_2} = \frac{1}{R_1}\frac{\partial\sqrt{G}}{\partial u},$$

he obtained the following equation in t

$$(2) \qquad \frac{\partial^2 \tan^{-1} t}{\partial v^2} + \frac{\partial \tan^{-1} t}{\partial u^2} = \frac{1-t^2}{1+t^2}\cdot\frac{1}{2g^2},$$

and the equations for E and G,

$$(3) \qquad \sqrt{E} = \frac{1}{\sqrt{2}}\frac{1+t}{\sqrt{(1+t^2)}}, \qquad \sqrt{G} = \frac{1}{\sqrt{2}}\frac{1-t}{\sqrt{(1+t^2)}}.$$

Denoting by σ the angle the plane $u =$ a constant makes with the surface, since

(4) $$\frac{R_1 R_2}{\sqrt{EG}} \frac{\partial}{\partial u} \left(\frac{\sqrt{G}}{R_2} \right) = -\operatorname{ctn} \sigma = \text{a function of } u \text{ alone},$$

he substituted the new variable

$$u_1 = \frac{1}{g} \int \operatorname{ctn} \sigma du .$$

He obtained from (1), (3) and (4) a differential equation for t. Integrating this equation and substituting the value thus obtained for t in equation (2), he differentiated the resulting equation twice with respect to u and arrived at a differential equation for v. The integral of this equation

$$\frac{1}{\sin^2 \sigma} = A \cosh 2u_1 + B \sinh 2u_1 + C,$$

contains three arbitrary constants. The form of the surface depends upon the value of these constants and the relation that exists between them.

Enneper concluded there was no loss of generality, in putting $B = 0$, and investigated accordingly. Later Kuen[85] showed that thereby a class of surfaces was overlooked.

After considerable reductions in which the two equations,

$$\left(g \frac{dv_1}{dv} \right)^2 = C - A \cosh 2v_1 \qquad (v_1 = \text{function of } v \text{ alone}),$$

$$\left(g \frac{du_1}{du} \right)^2 = C + A \cosh 2u_1 - 1,$$

formed a chief element, he proved that the planes of the lines of curvature $u =$ a constant meet in a straight line, and that the lines of curvature $v =$ a constant lie on spheres that cut the surface orthogonally and whose centers lie on the straight line.

In choosing the fixed line as the axis of z and representing by ϕ the angle between the intersection of the plane with the xy plane and the axis of x, he defined the surface by the following three equations,

$$x \sin \phi - y \cos \phi = 0 ,$$

$$(x \cos \phi + y \sin \phi) \cdot \frac{d\phi}{du} = - \frac{\sin \sigma}{\cosh (u_1 + v_1)},$$

$$\frac{1}{g} \int (C - A \cosh 2 v_1) \, dv - g \tanh (u_1 + v_1) \frac{dv_1}{dv} = \frac{z}{\sin^2 \sigma} \frac{\partial \phi}{\partial u},$$

where ϕ satisfies the equation

$$\left(\frac{d\phi}{du} \right)^2 = \frac{C^2 - A^2}{g^2} \sin^4 \sigma .$$

A detailed study of Enneper's surfaces was made by Bockwoldt [58] in 1874 for surfaces of constant positive curvature and by Lenz [66] in 1879 for surfaces of constant negative curvature in which the coördinates of points on the surface are expressed in elliptic functions of the two parameters.

There is one case and it was considered by Enneper in which the surface is expressed through the elementary functions, namely, when σ is constant and B as before is zero. He then took

$$C = \frac{1}{\sin^2 \sigma}$$

and the equation of the surface becomes

$$z = g \cos\sigma \tan^{-1}\frac{y}{x} + \sqrt{g^2 \sin^2\sigma - x^2 - y^2}$$
$$- \tfrac{1}{2} g \sin\sigma \log\left(\frac{g\sin\sigma + \sqrt{g^2\sin^2\sigma - x^2 - y^2}}{g\sin\sigma - \sqrt{g^2\sin^2\sigma - x^2 - y^2}}\right),$$

an equation which shows that the surface is generated by a tractrix whose vertex describes a helix on a right circular cylinder. This is Dini's helicoidal surface and it is thus found to occupy a special position in the Enneper's surfaces.

Kuen [85] in an interesting paper in 1884 set forth in a clear light the relations between the surfaces determined by Enneper and those which could be derived from the three surfaces of rotation by Bianchi's method for deriving one pseudospherical surface from another when the geodesic curves with reference to which they are derived meet at a point at infinity.*

6. Geodesic lines and their orthogonal trajectories were the only curves considered on pseudospherical surfaces until 1870 when Enneper [41] began to write on asymptotic curves. Since real asymptotic curves cannot exist except on surfaces of negative curvature, Enneper began his investigations on the subject for these surfaces only, afterwards supposing the curvature to be constant as well as negative. Asymptotic curves on pseudospherical surfaces possess peculiar properties that render them important in the infinitesimal deformation of a surface and in the deriving of a new pseudospherical surface from one that is known. For the latter operation it is important to know the expression for the linear element of the surface referred to asymptotic curves. Enneper found this expression by inserting in the Codazzi-Mainardi fundamental equations

$$\frac{\partial}{\partial v}\left(\frac{D}{\sqrt{EG-F^2}}\right) - \frac{\partial}{\partial u}\left(\frac{D'}{\sqrt{EG-F^2}}\right) + \frac{E\frac{\partial G}{\partial v} + F\frac{\partial G}{\partial u} - 2F\frac{\partial F}{\partial v}}{2(EG-F^2)} \cdot \frac{D}{\sqrt{EG-F^2}}$$

* Page 56.

$$-\frac{E\frac{\partial G}{\partial u}-F\frac{\partial E}{\partial v}}{(EG-F^2)}\cdot\frac{D'}{\sqrt{EG-F^2}}+\frac{\left(2E\frac{\partial F}{\partial u}-F\frac{\partial E}{\partial u}-E\frac{\partial E}{\partial v}\right)}{2(EG-F^2)}\cdot\frac{D''}{\sqrt{EG-F^2}}=0,$$

$$\frac{\partial}{\partial u}\left(\frac{D''}{\sqrt{EG-F^2}}\right)-\frac{\partial}{\partial v}\left(\frac{D'}{\sqrt{EG-F^2}}\right)+\frac{2G\frac{\partial F}{\partial v}-F\frac{\partial G}{\partial v}-G\frac{\partial G}{\partial u}}{2(EG-F^2)}\cdot\frac{D}{\sqrt{EG-F^2}}$$

$$-\frac{G\frac{\partial E}{\partial v}-F\frac{\partial G}{\partial u}}{(EG-F^2)}\cdot\frac{D'}{\sqrt{EG-F^2}}+\frac{\left(G\frac{\partial E}{\partial u}+F\frac{\partial E}{\partial v}-2F\frac{\partial F}{\partial u}\right)}{2(EG-F^2)}\cdot\frac{D''}{\sqrt{EG-F^2}}=0,$$

the values $D=0$, $D''=0$ and $K=-1/R^2=a$ constant, which are the conditions that the lines $u=a$ constant and $v=a$ constant shall be asymptotic curves on a surface with curvature $-1/R^2$. He thus reduced the equations to

$$G\frac{\partial E}{\partial v}-F\frac{\partial G}{\partial u}=0,\qquad E\frac{\partial G}{\partial u}-F\frac{\partial E}{\partial v}=0,$$

from which he saw that E is a function of u alone and G is a function of v alone, so that the expression for the linear element may be written

$$ds^2=du^2+2Fdudv+dv^2$$

and the equation for the curvature K becomes

$$K=-\frac{1}{R^2}=-\frac{1}{\sin 2\omega}\frac{\partial^2 2\omega}{\partial u\,\partial v}$$

where $2\omega=$ the angle between the asymptotic curves and $F=\cos 2\omega$.

Enneper discovered the famous theorem known as Enneper's theorem, that the square of the radius of torsion of an asymptotic curve at every point is equal to the product of the principal radii of curvature of the surface at that point, with the minus sign placed before it. In proving this theorem he obtained the two following equations for the curvature $1/\rho_v$ and the torsion $1/r_v$ of the asymptotic curve, $v=$ a constant,

$$\frac{1}{\rho_v}=\frac{\frac{\partial}{\partial u}\left(\frac{F}{\sqrt{E}}\right)-\frac{\partial\sqrt{E}}{\partial v}}{\sqrt{EG-F^2}},\qquad \frac{1}{r_v}=\frac{D'}{\sqrt{EG-F^2}},$$

the first of which shows that its geodesic curvature is equal to its curvature, and the second that its torsion squared is equal in value but opposite in sign to the curvature of the surface, and is consequently constant when the curvature of the surface is constant.

Enneper also remarked that if one surface is applied on another, one system

only of asymptotic curves on the first surface can by any possibility pass over into asymptotic curves on the second surface, as, for example, the generators of a skew surface.

In the same year, Dini[47] made a study of aymptotic curves. Supposing the surface to be represented upon a sphere after the method of Gauss, he denoted its linear element referred to arbitrary parameters u and v by

$$ds^2 = Edu^2 + 2Fdu\,dv + Gdv^2,$$

and the spherical image of its linear element by

$$ds^2 = E'du^2 + 2F'du\,dv + G'dv^2.$$

He derived the Codazzi-Mainardi equations for the coefficients E', F', G', and for the coefficients D, D', D'', of the second fundamental differential expression for the surface and introducing the conditions necessary in order that the parametric lines on the sphere should represent the asymptotic curves of a surface of negative curvature $-\mu^2$, he reduced these equations to

$$\frac{\partial \frac{E'}{\mu}}{\partial v} = F'\,\frac{\partial}{\partial u}\left(\frac{1}{\mu}\right), \qquad \frac{\partial \frac{G'}{\mu}}{\partial u} = F'\,\frac{\partial}{\partial v}\left(\frac{1}{\mu}\right),$$

and to

$$\frac{\partial E'}{\partial v} = 0, \qquad \frac{\partial G'}{\partial u} = 0,$$

when μ is a constant.

His expression for the spherical representation of the linear element is therefore,

$$ds'^2 = du^2 + 2F'\,du\,dv + dv^2$$

and for the linear element of the surface itself it is

$$ds^2 = \frac{du^2 - 2F'du\,dv + dv^2}{\mu^2},$$

for in an earlier paper he had remarked that the arc lengths of asymptotic curves are always proportional to the arc lengths of their spherical image in the ratio of the curvature of the sphere to the curvature of the surface and that the angle between two asymptotic curves on the surface is the supplement of the angle between the two lines that represent them on the sphere.

Dini was the first to observe from the form of this expression that "asymptotic curves divide a surface into infinitely small lozenges." Hazzidakis[59] went a step further than Dini and found the area of one of these lozenges to be $A + B + C + D - 2\pi$ where A, B, C, D represent its four angles. He obtained this value by integrating along its boundary the equation for the area of the quadrilateral

$$\iint \sin 2\omega du\, dv,$$

where $\sin 2\omega$ is given by the equation for the measure of curvature K,

$$K = -\frac{1}{\sin 2\omega}\frac{\partial^2 2\omega}{\partial u\,\partial v} = -1.$$

Voss[71] approached the subject of asymptotic curves from the consideration of equi-distant curves. He gave that name to a system of curves on a surface which form a net-work of quadrilaterals whose opposite side are equal. His expression for the linear element of the surface, when $u =$ a constant and $v =$ a constant represent these lines, becomes

$$ds^2 = du^2 + 2 \cos 2\omega du\, dv + dv^2,$$

where $2\omega =$ the obtuse angle of a quadrilateral. To find the surface for which the equi-distant curves are asymptotic lines he made the necessary substitutions in the Codazzi-Mainardi equations

$$\begin{aligned} D &= 0, & D'' &= 0, \\ E &= 1, & G &= 1, \end{aligned}$$

and reduced them to two partial differential equations

$$\frac{\partial}{\partial u}\left(\frac{D'}{\sqrt{EG - F^2}}\right) = 0, \qquad \frac{\partial}{\partial v}\left(\frac{D'}{\sqrt{EG - F^2}}\right) = 0,$$

whose common solution is

$$\frac{D'}{\sqrt{1 - F^2}} = \text{a constant.}$$

This expression denotes the measure of curvature of the surface with the negative sign, so that Voss thus proved that only upon surfaces of constant negative curvature can a system of equi-distant curves be composed of asymptotic lines.

Voss also found the characteristic equation for surfaces of constant negative curvature $-1/R^2$

$$\frac{\partial^2 2\omega}{\partial u \partial v} = \frac{\sin 2\omega}{R^2}$$

by deforming the meridian and parallel curves of the pseudosphere into equidistant curves.

The equation for an asymptotic curve on a pseudosphere was derived by Beltrami[18] in 1872. In that year he published a paper whose title, "On the Surface of Rotation that serves as a Type for all Pseudospherical Surfaces," shows the nature of its contents. In order to obtain a set of equations for the surface he called the axis of rotation the axis of z, the plane of the maximum parallel circle the $x\,y$ plane, the angle measured on this plane that any meridian

makes with a fixed meridian the angle ϕ and the acute angle that the tangent to the meridian at any point makes with the axis of rotation the angle θ. His equations for the coördinates x, y, z of a point on the surface of curvature $-1/r^2$ are then

$$x = r \sin\theta \cos\phi, \qquad y = r \sin\theta \sin\phi, \qquad z = r\left(\log \operatorname{ctn}\frac{\theta}{2} - \cos\theta\right).$$

Every point on the surface is therefore determined by the value of ϕ and θ at that point. The expression for the linear element of the surface then becomes

$$ds^2 = r^2(\operatorname{ctn}^2\theta\, d\theta^2 + \sin^2\theta\, d\phi^2)$$

and may be transformed into the usual form

$$ds^2 = du^2 + r^2 e^{-2u/r} dv^2$$

by means of the equations

$$\sin\theta = e^{-u/r}, \qquad \phi = v,$$

while, if R_1 and R_2 denote the principal radii of curvature,

$$R_1 = -r \operatorname{ctn}\theta, \qquad R_2 = r\tan\theta.$$

Beltrami began the study of asymptotic curves from a very interesting point of view. He considered first a curve which he defined as "the curve of intersection of the surface with the tangent plane to it at the point $\theta = \theta_0$ of the meridian lying in the xz plane." He proved with respect to this curve that it has two branches going out from the point of tangency $(\theta = \theta_0)$ each of which makes an angle θ_0 with the tangent to the meridian curve and that when very small arc lengths of the osculating circles of those two branches measured from the branch point are revolved about the axis of z they will generate a surface differing by a quality of the fourth order only from the original surface. The two branches of the curve of intersection, since their planes of osculation are the tangent plane to the surface at the point $\theta = \theta_0$ will coincide respectively with the two asymptotic curves of the surface going through that point; accordingly Beltrami remarked that if two asymptotic lines be drawn through a point on a meridian curve they will each make an angle with the tangent to the meridian at that point that is equal to the angle which that same tangent makes with the axis of rotation. He built up the following series of propositions on this theorem:—

Since the equation for the sine of the angle ψ which any curve makes with the geodesic meridian curve $\theta = \theta_0$ is given by

$$\sin\psi = -\frac{\partial\sqrt{G}}{\partial u}\cdot\frac{dv}{ds} = +\frac{r\sin\theta d\phi}{ds}$$

and when the curve is an asymptotic cuve $\psi = \theta$, the arc length of an asymptotic curve is given by

$$ds = \pm\, r d\phi .$$

The integral of this equation

$$s = r(\phi - \phi_0)$$

shows that the arc length of the curve between the meridans $\phi = \phi$ and $\phi = \phi_0$ is equal to its orthogonal projection on the plane of the maximum geodesic circle and that each one of the infinite number of portions into which the length of the curve is divided by a meridian curve is equal to the circumference of the maximum parallel. If a linear element of surface be taken along an asymptotic curve, there results the equation,

$$ds^2 = r^2(\operatorname{ctn}^2\theta d\theta^2 + \sin^2\theta d\phi^2) = r^2 d\phi^2$$

or

$$\mp \frac{1}{\sin\theta} d\theta = d\phi$$

and the integral of this equation,

$$\log\tan\frac{\theta}{2} = \phi - \phi_0$$

or

$$\sin\theta\cosh(\phi - \phi_0) = 1\,,$$

gives the equation of an asymptotic curve that touches the maximum parallel at the point ϕ_0.

The part played by an asymptotic line in the infinitesimal deformation of surfaces was discovered by Jellet[12] in 1853, who gave the theorem that, when a surface is deformed infinitely little, one asymptotic curve may remain unchanged. He therefore called asymptotic lines "curves of flexion," and stated the proposition, "We can fix a curve of flexion without preventing the deformation of any finite portion of the surface." As only surfaces of negative curvature have real asymptotic curves, they are the only surfaces that can be bent while a curve on them is unchanged. This theorem was demonstrated by Lecornu[65] in 1880 and by Weingarten[95] in 1886.

In this connection, Weingarten introduced the idea of the bending invariant. He denoted by $\epsilon\bar{x}$, $\epsilon\bar{y}$, $\epsilon\bar{z}$ the infinitesimal increments that each coördinate x, y, z of a point receives when the surface on which it lies is bent infinitely little, and by x', y', z' the coördinates of the same point after the surface is bent so that

$$x' = x + \epsilon\bar{x}, \qquad y' = y + \epsilon\bar{y}, \qquad z = z + \epsilon\bar{z}.$$

Then assuming the expression for the linear element of the surface to be

$$ds^2 = Edu^2 + 2Fdu\,dv + Gdv^2,$$

he defined the bending invariant ϕ by means of the equation

$$\phi = -\frac{1}{\sqrt{EG - F^2}}\left(\sum \frac{\partial \bar{x}}{\partial u}\cdot\frac{\partial x}{\partial v} - \sum \frac{\partial \bar{x}}{\partial v}\cdot\frac{\partial x}{\partial u}\right).$$

He, moreover, showed that since the linear element of the surface remains unchanged when the surface is bent, and ϵ is so small that when raised to the second power it may be neglected, it must happen that

$$\Sigma dx\, d\bar{x} = 0\,,$$

or $\bar{x}$, $\bar{y}$, $\bar{z}$ may be regarded as the coördinates of a point on the second surface that corresponds to the first by the orthogonality of its elements.

A second theorem of bending that relates to surfaces of negative curvature is that when two surfaces are associated, that is, when the bending invariant of the one at every point is equal to the distance of the tangential plane at corresponding points of the other from the origin, the total curvature of one of the surfaces must be negative. The discovery both of the existence of such pairs of surfaces and of the theorem concerning them is the work of Bianchi.[139] The associated surface of pseudospherical surfaces have been studied within the past few years by Cosserat,[130, 131] Guichard [114, 119] and Voss.

The use of asymptotic lines in the transformation of one pseudospherical surface into another will be considered later. It is necessary here to turn to the development of the geometry of lines on these surfaces.

7. While the mathematicians of France, Italy and Germany were discovering the various properties of surfaces of negative constant curvature and adding from time to time to the development of their theory so as to make them take an important position in the class of surfaces of constant curvature, considering them merely as a necessary adjunct to the completion of the study of that class, another topic of more general interest was attracting the attention of men all over the world. This matter was none other than the recognition that Euclid's fifth postulate, equivalent to the statement that only one line parallel to a fixed line can be drawn through a point, is not capable of demonstration from the preceding hypotheses.

Gauss[2] among others gave some study to the subject and recognized in connection with it the existence of a new geometry, which he called the non-Euclidian, and which he distinguished from the Euclidian by the essential characteristic that in it there is never any similitude in the figures without equality.

Gauss never published a complete exposition of his theories, but referred to them occasionally in various papers since published in the Göttingen edition of his works and in his correspondence with Schumacher. It is in a letter to the latter that he gave the now familiar expression for the semi-circumference of a circle with radius r in the non-Euclidian geometry,

$$\frac{\pi K}{2}\left(e^{\frac{r}{K}} - e^{-\frac{r}{K}}\right) \qquad (K = \text{a constant})$$

and remarked that for the Euclidian geometry K becomes infinitely great, but Gauss' contributions to the new geometry were slight in comparison to those of Lobatchewsky.

In 1831 Lobatchewsky [4] produced a pamphlet on the theory of parallel lines, of which Gauss said in another letter to Schumacher, "I have found in the work of Lobatchewsky no surface new to me, but the statement is entirely different from that I had contemplated."

In this pamphlet, Lobatchewsky set forth a whole imaginary geometry based on the hypothesis that two lines parallel to a third may be drawn through a point and demonstrated a series of propositions analogous to those of the Euclidian geometry. In 1854 Riemann [13] wrote his renowned Habilitationschrift in which he introduced for hyperspace of any dimensions the idea of three kinds of constant curvature, positive, zero and negative. In particular he considered two-fold space and stated that all surfaces of positive curvature are developable upon a sphere and all those of zero curvature upon a cylinder. "Surfaces of negative curvature," he said, "will touch the cylinder externally and be found like the inner position (towards the axis) of the surface of a ring." He made the further statement that "the surfaces with positive curvature can always be so formed that figures may also be moved arbitrarily about upon them without bending, namely they may be formed into sphere surfaces; but not those with negative curvature." He thus suggested the idea of a geometry on surfaces of constant negative curvature as opposed to spherical and plane Euclidian geometry. No mathematician, however, connected the new geometry of Lobatchewsky with the geometry of pseudospherical surfaces until Beltrami [32] wrote his essay on the non-Euclidian geometry in which he showed analytically that all the propositions and theorems of the new geometry can be realized by means of figures lying upon such a surface.

His method of proof was based upon such a choice of parameters u and v that a linear equation between them represents a geodesic line on the surface. Consequently the surface may be represented upon a plane in such a way that its geodesic lines become straight lines. Beltrami [28] wrote a paper in 1865 on this representation showing that it is possible only for surfaces with constant curvature, and that it is analogous to the central projection of a sphere together with its linear substitutions.

He represented the geodesic lines by

$$au + bv + c = 0$$

and the straight line on the plane by

$$ax + by + c = 0$$

so that the equations for transforming the one into the other are

$$u = x, \qquad v = y,$$

and the plane and surface correspond to each other point to point.

Since the general differential equation of any geodesic curve is

$$2(du\,d^2v - d^2u\,dv) = \frac{1}{2(EG - F^2)}$$

$$\left\{\begin{array}{l} \left(F\frac{\partial E}{\partial u} - 2E\frac{\partial F}{\partial u} + E\frac{\partial E}{\partial v}\right)du^3 - \left(2E\frac{\partial G}{\partial u} - G\frac{\partial E}{\partial u} - 3F\frac{\partial E}{\partial v} + 2F\frac{\partial F}{\partial u}\right)du^2\,dv \\ -\left(-2G\frac{\partial E}{\partial v} + E\frac{\partial G}{\partial v} + 3F\frac{\partial G}{\partial u} - 2F\frac{\partial F}{\partial v}\right)du\,dv^2 + \left(2G\frac{\partial F}{\partial v} - F\frac{\partial G}{\partial v} - G\frac{\partial G}{\partial u}\right)dv^3 \end{array}\right\}$$

and in the case considered this must become

$$du\,d^2v - d^2u\,dv = 0,$$

he saw that the coefficient of each term of the right-hand member of the equation must vanish identically so that the reduction furnishes a set of equations whose integrals give for E, F and G the values

$$E = \frac{R^2(v^2 + a^2)}{(u^2 + v^2 + a^2)^2}, \qquad F = \frac{-R^2uv}{(u^2 + v^2 + a^2)^2}, \qquad G = \frac{R^2(u^2 + a^2)}{(u^2 + v^2 + a^2)^2},$$

where $1/R^2$ is the curvature of the surface and a is an arbitrary constant.

He was thus able to write down at once the expression for the linear element of a surface with constant positive curvature

$$ds^2 = \frac{R^2\left((a^2 + v^2)du^2 - 2uv\,du\,dv + (a^2 + u^2)dv^2\right)}{(u^2 + v^2 + a^2)^2}$$

which he changed into

$$ds^2 = \frac{R^2\left((a^2 - v^2)du^2 + 2uv\,du\,dv + (a^2 - u^2)dv^2\right)}{(a^2 - u^2 - v^2)^2}$$

for pseudospherical surfaces by writing $-R^2$ and $-a^2$ for R^2 and a^2.

This expression for the linear element was his starting point for his investigations on the non-Euclidian geometry in 1868 and from it he developed other properties peculiar to the surface and to Lobatchewsky's imaginary plane. He observed that if θ be the angle between two lines $u =$ a constant and $v =$ a constant that

$$\cos\theta = \frac{uv}{\sqrt{(u^2 - a^2)(v^2 - a^2)}}, \quad \sin\theta = \frac{a\sqrt{a^2 - u^2 - v^2}}{\sqrt{(u^2 - a^2)(v^2 - a^2)}}.$$

By using polar coördinates r and ϕ he found a second expression for the linear element for the surface

$$ds^2 = R^2\left\{\left(\frac{adr}{a^2 - r^2}\right)^2 + \frac{r^2 d\phi^2}{a^2 - r^2}\right\}.$$

From this he derived equations for the length ρ of a geodesic line $\phi =$ a constant and for the arc σ of a geodesic circle $r =$ a constant, or as he called it, a geodesic circumference,

$$\rho = \frac{R}{2}\log\frac{a + \sqrt{u^2 + v^2}}{a - \sqrt{u^2 + v^2}} = \frac{R}{2}\log\frac{a + r}{a - r},$$

$$r = a\tanh\frac{\rho}{R},$$

$$\sigma = \phi R\sinh\frac{\rho}{R}.$$

His expression for the circumference of a geodesic circle is therefore similar to to the one found by Gauss,

$$\pi R(e^{\rho/R} - e^{-\rho/R}).$$

From these equations he saw that the curve whose equation is

$$u^2 + v^2 = a^2,$$

bounds the region of real values. He remarked that when the surface is represented upon a plane this curve becomes a circle which he called the limiting circle because all the points corresponding to real points on the surface lie within it, all those corresponding to the ideal or imaginary points on the surface lie without it and the points on its circumference correspond to infinitely far off points on the surface. He also showed that the geodesic lines of the surface become chords of this circle and that, since two points fully determine a chord, two points will determine a geodesic line.

From the equation for θ he further observed the nature of the parametric lines on the surface, that they consist of two systems of geodesic lines which are so related to each other that the two fundamental lines $v = 0$, $v = 0$, meet at right angles at the origin while the coördinate lines $u =$ a constant are orthogonal to $v = 0$ and the coördinate lines $v =$ a constant are orthogonal to $u = 0$.

He showed by a rigorous proof that any two lines that cut each other orthogonally may be chosen for the fundamental lines, instead of $u = 0$, $v = 0$, and that consequently any geodesic line may be made to coincide with any other and the surface superposed upon itself, for changing the pair of orthogonal geodesic lines intersecting at the origin into any other set of orthogonal geodesic lines through any other point does not alter the form of the expression for the linear element.

These two characteristics, the superposability of the surface upon itself and the determination of a geodesic line by two points, Beltrami called the "funda-

mental criteria of elementary geometry," and since they belong equally to pseudospherical surfaces and to the Lobatchewskian plane, he said: "It becomes evident that the theorems of the plane non-Euclidian geometry exist unconditionally for all the surfaces of constant negative curvature."

The keystone of the non-Euclidian geometry is the proposition that two straight lines can be drawn through any fixed point parallel to a given straight line. Beltrami proved this proposition by means of his geodesic representation of the pseudospherical surface on the plane in the following manner: first it is necessary to show that the angle between two geodesic lines intersecting at a real finite point on the surface is never 0 nor π, but that the angle may be 0 or π when the curves intersect at a point of infinity. If this angle is represented by ψ and the angle on the plane between two chords corresponding to the geodesic lines be represented by ψ' and the angles which the chords make with the axis of X by μ and ν respectively, ψ and ψ' are related by means of the equation

$$\tan \psi = \frac{a(\sqrt{a^2 - u^2 - v^2}) \sin \psi'}{a^2 \cos \psi' - (v \cos \mu - u \sin \mu)(v \cos \nu - u \sin \nu)},$$

the right-hand member of which can only be zero when $u^2 + v^2 = a^2$, that is when the two chords meet on the perimeter of the circle, consequently the angle ψ is 0 only when the geodesic curves meet at a point at infinity which corresponds with the point of intersection of the chords on the perimeter of the limiting circle.

When a given geodesic and a given point on the surface are represented by a chord of the limiting circle on the plane and a point within its perimeter, two of the chords drawn through that point will meet the first chord at its extremities on the circumference of the circle, therefore, the two geodesic lines which correspond to the two chords will meet the given geodesic line at infinity, making with it an angle 0 and they will therefore both be parallel to it though drawn through one point.

Following the line of thought laid down by Lobatchewsky, Beltrami next defined the angle of parallelism Π as half the angle between the two geodesic lines drawn through a fixed point parallel to a given geodesic line. To determine $\tan \Pi$ he constructed the corresponding angle and lines on the plane. He chose the center of the limiting circle to represent the fixed point and the line corresponding to the geodesic bisector of the angle of parallelism for the axis of x so that the coördinates of the extremities of the chord representing the given geodesic line are (x, y) and $(x, -y)$. He could then write

$$\tan \Pi = \frac{y}{x} = \frac{\sqrt{a^2 - x^2}}{x}$$

and from the equation

$$r = \sqrt{u^2 + v^2} = a \tanh \frac{\rho}{R} \qquad (\rho = \text{length of geodesio bisector}),$$

he could obtain for x along the axis, $y = 0$, the value

$$x = a \tanh \frac{\rho}{R},$$

so that on the surface

$$\tan \Pi = \frac{1}{\sinh \frac{\rho}{R}},$$

a form equivalent to the one given by Lobatchewsky.

By means of the above equation he was able to express Minding's equation for the angle of a geodesic triangle in terms of the angles of parallelism of the sides and thus obtain the fundamental equation of the non-Euclidian trigonometry

$$\cos A \cos \Pi\,(b) \cos \Pi\,(c) + \frac{\sin \Pi\,(b) \sin \Pi\,(c)}{\sin \Pi\,(a)} = 1$$

where a, b, c are the sides and A, B, C the angles of the triangle.

Finally he found the area of a triangle to be proportional to its spherical deficiency, a fact which results from Gauss' theorem that its total curvature is equal to the sum of its angles minus π.

From the theorems of pure geometry Beltrami returned to the subject of the three different forms of the pseudospherical surface of rotation. He first found the equation for a geodesic circle, or as he called it a geodesic circumference, whose center is the point u, v, and whose radius is ρ, to be

$$\frac{a^2 - uu_0 - vv_0}{\sqrt{(a^2 - u^2 - v^2)(a^2 - u_0^2 - v_0^2)}} = \cosh \frac{\rho}{R}$$

and by means of it he deduced that the expression for the linear element assumes one of the three different forms given by Dini,

$$ds^2 = (du^2 + \sinh^2 \frac{u}{R} dv^2),$$

$$ds^2 = (du^2 + e^{2u/R} dv^2),$$

$$ds^2 = (du^2 + \cosh^2 \frac{u}{R} dv^2),$$

according as to whether the centers of the geodesic circumferences chosen for one family of parametric curves are real, at infinity or ideal, that is whether on the plane, the corresponding points lie within the limiting circle, on its perimeter or entirely without it.

He also remarked upon the peculiar properties of the three types of geodesic

circumferences, how, those of the third type with a common center are parallel to a geodesic curve, a property which belongs to all geodesic circles on a sphere, but which belongs only to geodesic circles with an ideal center on a pseudospherical surface; how a geodesic circle of the second type is identical with what is known as the limiting circle or horicycle of Lobatchewsky, that is, a curved line such that all the perpendiculars erected at the middle point of its chords are parallel to each other; how the geodesic lines orthogonal to a family of geodesic circles of the first type go through a common point usually chosen for the origin ($u = v = 0$).

In this same paper, in speaking of the three types of surfaces of rotation which correspond to the above three forms of the linear element, he remarked that in the actual application of a surface of rotation of the first type upon a pseudospherical surface of a different form, it is necessary to make a slit in the surface from the point of intersection ($u = v = 0$) of the meridian curves in order to apply the "pseudospherical cap" about the point ($u = v = 0$) upon the second surface. He went on to observe that surfaces of rotation of the second or third type have each a minimum parallel circle, that for the last named surface, this minimum circle is the geodesic curve to which all the other parallel circles are parallel and that at equal distances from it, on either side, lie two maximum parallel circles between which lies the real part of the surface and that when a pseudospherical surface is applied upon a surface of rotation of the second or third type it may be wrapped about it an infinite number of times. These properties though evident from the drawings of pseudospherical surfaces at the end of Minding's memoir in volume XIX of Crelle's Journal, were not described by him nor were they spoken of in any of the papers previously mentioned.

8. Beltrami[48] wrote a paper in 1872 devoted exclusively to the subject of the pseudosphere, making therein a particular study of its asymptotic and geodesic curves. His theorems on geodesic lines, following in natural order after his remarks in regard to these curves in his "Essay on the Interpretation of the Non-Euclidian Geometry," will now be considered.

If the expression for the linear element of the pseudosphere of curvature $-1/r^2$ be written in the usual form

$$ds^2 = d\sigma^2 + e^{-2\sigma/r}\,d\phi^2$$

where the parameter σ represents the arc length of any meridian curve and the parameter ϕ denotes the angle that that meridian makes with a fixed meridian measured on the plane of the maximum parallel, the general equation for the radius of geodesic curvature of a curve becomes

$$\rho = 2r\,\frac{\left[1 + \left(\dfrac{\partial \epsilon^{\sigma/r}}{\partial \phi}\right)^2\right]^{\frac{3}{2}}}{\dfrac{\partial^2 (\epsilon^{2\sigma/r} + \phi^2)}{\partial \phi^2}}.$$

Beltrami obtained the equation for a parallel circle.

$$a\left(e^{2\sigma/r} + \phi^2\right) + 2be^{\sigma/r} + 2c\phi + d = 0$$

by putting $\rho =$ a constant in this equation and then integrating. The denominator set equal to zero gave him the differential equation for a geodesic curve whose integral is

$$e^{2\sigma/r} + (\phi - b)^2 = (a + b^2). \qquad (a,\ b,\ c,\ d \text{ are constants.})$$

If the equation for a parallel circle be differentiated twice with respect to ϕ it will become

$$\frac{d^2\left(\epsilon^{2\sigma/r} + \phi^2\right)}{d\phi^2} = -\frac{b}{a}\frac{d^2e^{2\sigma/r}}{d\phi^2}$$

and the combination of this equation with the equation for the radius of geodesic curvature makes the latter assume the form

$$\frac{b\rho}{ar} = -\frac{\left[1 + \left(\frac{de^{\sigma/r}}{d\phi}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2e^{\sigma/r}}{d\phi^2}}.$$

The comparison of this equation with the ordinary one of Differential Calculus for the radius of curvature of a plane curve showed him that the two become identical if $\phi = x$ and $e^{\sigma/r} = y$. He saw, therefore, that the geodesic circles of the surface may be transformed into circles on a plane whose equation is

$$\left(y + \frac{b}{a}\right)^2 + \left(x + \frac{c}{a}\right)^2 = \frac{b^2 + c^2 - ad}{a^2} = \frac{b^2\rho^2}{a^2r^2},$$

and that the condition that these circles should have a center that is a real finite point, a point at infinity or an imaginary point may be expressed analytically by

$$\rho \lesseqqgtr r.$$

This projection, which is similar to the stereographic projection of a sphere, has since become very useful in the investigation of pseudospherical surfaces.

Beltrami's first method of projection or geodesic representation of a pseudospherical surface converts the geodesic lines of a surface into straight lines on the plane; his second method transforms the geodesic lines on the surface into circles on the plane.

Beltrami's second method of projection is conformal, and Busse[158] has shown in a recent doctor's dissertation that it is only surfaces of constant curvature that can be conformally projected upon a plane in such a way that their geodesic curves become right lines or the arcs of circles.

A very interesting deduction was made by Cayley[67] in 1884 from the theorems contained in Beltrami's two papers on pseudospherical geometry, namely,

that "the Lobatchewskian geometry is a geometry such as that of the imaginary spherical surface $X^2+Y^2+Z^2=-1$ (spoken of by Dini, page 18) and that the imaginary surface may be bent without extension or contraction into the real surface considered by Beltrami."

He remarked that this bending is an "imaginary process" for the points and lines on the first surface are imaginary and those on the second are real, while the angles and distances are real on both surfaces. He denoted the coördinates of a point on the imaginary sphere of curvature -1 by X, Y, Z, and the coördinates of a point on a pseudosphere of the same curvature by x, y, z. He was then able to transfer the linear element on the surface denoted by

$$ds^2 = dX^2 + dY^2 + dZ^2$$

into the linear elements of the pseudosphere represented by

$$ds^2 = dx^2 + dy^2 + dz^2,$$

by means of the three sets of equations

$$X = \frac{-i}{\sqrt{1-u^2-v^2}}, \qquad Y = \frac{u}{\sqrt{1-u^2-v^2}}, \qquad Z = \frac{v}{\sqrt{1-u^2-v^2}},$$

$$\sin\theta = \frac{1-u}{\sqrt{1-u^2-v^2}} = e^{-\sigma}, \qquad \phi = \frac{v}{1-u},$$

$$x = \cos\phi\sin\theta, \qquad y = \sin\phi\sin\theta, \qquad z = \log\operatorname{ctn}\frac{\theta}{2} - \cos\theta,$$

where u and v are Beltrami's parameters which, when linearly connected, represent a geodesic curve.

Having established the fact that the imaginary sphere is transformable into a real pseudosphere, Cayley proceeded to consider the geodesic curves on the first surface. The equation of a geodesic curve on the imaginary sphere may be written in the form similar to that of a geodesic curve on a real sphere

$$aX + bY + cZ = 0 \qquad (a,\ b,\ c, = \text{constants}),$$

but Cayley observed that "since for a point corresponding to a real point of a pseudosphere X is a pure imaginary, and Y and Z are real, we see that for a geodesic corresponding to a real geodesic of the pseudosphere X must be a pure imaginary and X and Z real. In order to have all the coefficients real he therefore made the substitutions

$$P = iX - Y, \qquad Q = iX + Y,$$

by means of which the equation becomes

$$(-\tfrac{1}{2}ia - \tfrac{1}{2}b)P + (-\tfrac{1}{2}ia + \tfrac{1}{2}b)Q + cZ = 0,$$

or

$$AP + BQ + CZ = 0,$$

where A, B and C are all real. Applying the same equations of transformation to this equation as he had formerly applied to the imaginary sphere to deform it into a pseudosphere he found that the equation assumes the form

$$A + B(e^{2\sigma} + \phi^2) + C\phi = 0\,,$$

which is the form obtained by Beltrami for the geodesic curves on a pseudosphere.

Thus Cayley proved that the imaginary surface and the real surface are so related to each other that to every point and to every geodesic line of the one there corresponds a point and a geodesic line of the other.

Cayley applied the same method of projection on the plane of the greatest parallel to the case of a geodesic line that cuts a meridian curve at right angles, as Beltrami had applied to asymptotic curves. By tracing the course of the projected line he saw that it continues to cut at right angles the radius of the maximum circle into which the meridian is projected, that in the neighborhood of the circumference of the circle it is almost a straight line and that the further away the point of intersection of the meridian curve with the geodesic line on the surface lies from the plane of the unit circle, the nearer the projection of the line approaches the center of the circle and the more curved it becomes, while the circle itself is an envelope of geodesic lines.

The question as to whether Beltrami's geodesic projection of a pseudospherical surface on a plane may represent the whole plane of Lobatchewsky's geometry was asked by Hilbert[182] and answered by him in the negative, for he proved that it is impossible to construct an analytic surface with constant negative curvature that contains no singularities. First he assumed that such a surface can be constructed, and showed that in that case it will be completely covered by a net-work composed of two families of asymptotic lines, for he proved that no one of these lines ever intersects one of the curves of the family to which it does not belong more than once and never intersects a member of its own family, and that they have no double points or singularities of any kind. He then saw that the surface can be regarded as bounded by four of these asymptotic lines no matter what its extent may be and that by Dini's theorem its area will never be greater than 2π. On the other hand he recalled Gauss' expression for the area of a geodesic circle with radius ρ on a surface of curvature $-1/R^2$,

$$R\pi(e^{\rho/R} - e^{-\rho/R}),$$

and saw that, if he supposed the surface to be bounded by such a circle with a radius indefinitely great, its area must be greater than 2π. Such an inconsistency between the two methods of measurement showed him that there must be singularities somewhere on the surface, and that therefore the projection of such an analytic surface does not represent the whole of Lobatschewsky's plane.

9. It remained for Klein[55] to reconcile these two geometries, the Pseudo-spherical geometry of Beltrami and the non-Euclidian geometry of Lobatchewsky with still a third, the Metrical geometry of Cayley.

Cayley[14] first originated this geometry in 1859, as a result of his studies on the projective properties of points, lines and planes. In this connection, he considered the distance between two points (x_1, y_1) and (x_2, y_2) on a plane as denoted by the formula

$$\cos^{-1}\frac{x_1x_2+y_1y_2}{\sqrt{x_1^2+y_1^2}\sqrt{x_2^2+y_2^2}}$$

and between two points (x_1, y_1, z_1), (x_2, y_2, z_2) on a sphere by the formula

$$\cos^{-1}\frac{x_1x_2+y_1y_2+z_1z_2}{\sqrt{x_1^2+y_1^2+z_1^2}\sqrt{x_2^2+y_2^2+z_2^2}}$$

where $x_1, y_1, z_1, x_2, y_2, z_2$, are ordinary rectilinear coördinates. Cayley observed that the first formula might represent the angle between the polars of the points with respect to a conic whose equation is

$$x^2+y^2=0$$

and the second, the angle between the polars of the points with respect to a spherical conic whose equation is

$$x^2+y^2+z^2=0.$$

Therefore in order to measure the distance between any two points on a plane, he assumed an imaginary conic which he called the "absolute" and formed the expression

$$\cos^{-1}\frac{(a, b, c\between x_1, y_1, z_1\between x_2, y_2, z_2)}{\sqrt{(a, b, c\between x_1, y_1, z_1)^2}\sqrt{(a, b, c\between x_2, y_2, z_2)^2}}$$

where x_1, y_1, z_1 and x_2, y_2, z_2 are the homogeneous coördinates of the points and

$$(a, b, c\between x, y, z)^2 = ax^2+by^2+cz^2=0,$$

is the equation of the conic.

He observed the fact that the two points together with the points of intersection of their binding line with the absolute are in involution and that two systems in involution are homograpically related, He also discovered that if line coördinates are used instead of point coördinates exactly the same formula will measure the angle between two lines, and that, in that case, these lines and the tangents drawn to the absolute from their point of intersection are in involution.

Cayley himself never applied his theories to the case of pseudospherical surfaces but Klein, perceiving that if he assumed a general formula.

$$2iC\cos^{-1}\frac{(a, b, c\between x_1, y_1, z_1\between x_2, y_2, z_2)}{\sqrt{(a, b, c\between x_1, y_1, z_1)^2}\sqrt{(a, b, c\between x_2, y_2, z_2)^2}},$$

for the measure of distance between two points on a plane, he could derive from it Cayley's expression by putting for C the value $-i/2$ and the expression of the Euclidian geometry by letting C become infinitely great introduced as a third value for C a real finite quantity, and thus obtained an expression that satisfies the requirements of the theorems of the non-Euclidian geometry.

Klein denoted the absolute in homogeneous point coördinates by

$$\Omega = 0,$$

so that the general expression for the distance between two points x and y becomes

$$2iC\cos^{-1}\frac{\Omega_{xy}}{\sqrt{\Omega_{xx}\Omega_{yy}}},$$

where Ω_{xx} and Ω_{yy} are the expressions which result when the coördinates x_1, x_2, x_3, of the point x or the coördinates y_1, y_2, y_3 of the point y are set in Ω, and Ω_{xy} is the consequence of putting the coördinates of x in the polar of y or conversely.

He changed this general expression into the equivalent form

$$C\log\frac{\Omega_{xy}+\sqrt{\Omega_{xy}^2-\Omega_{xx}\Omega_{yy}}}{\Omega_{xy}-\sqrt{\Omega_{xy}^2-\Omega_{xx}\Omega_{yy}}},$$

and observed that the expression under the sign of the logarithm is the anharmonic ratio formed by the two points x and y and the points of intersection with the absolute of the line joining them.

He obtained a similar expression for the distance between two lines represented by u and v, namely

$$C'\log\frac{\Phi_{uv}+\sqrt{\Phi_{uv}^2-\Phi_{uu}\Phi_{vv}}}{\Phi_{uv}-\sqrt{\Phi_{uv}^2-\Phi_{uu}\Phi_{vv}}},$$

when $\Phi = 0$ is the equation of the absolute in homogeneous line coördinates. He saw that both expressions under the sign of the logarithm are anharmonic ratios, the first formed of four points, the second of four lines, each of which, according to Cayley, belongs to a system in involution, and that therefore every point in a line except the points of its intersection with the absolute may be linearly transformed in every other point and every ray in a pencil, except the two tangent to the absolute, may be linearly transformed into every other ray.

Klein investigated the nature of the absolute and discovered its characteristic properties; first, that since for an imaginary value of C it is imaginary, for a

real value of C it must be real, and that, in that case, since only real distances are considered, the anharmonic ratio is positive and all real points lie within its circumference; second, that it lies at infinity, for if a conic is assumed to be a circle with x as its center and y a point on its circumference, its radius will be by Cayley's formula equal to

$$2iC\cos^{-1}\frac{\Omega^2_{xy}}{\Omega_{xx}\Omega_{yy}},$$

and will become infinitely great when y lies on the circumference of the conic, $\Omega_{yy}=0$; third, that it is impossible to determine the region outside of the absolute, "the ideal region," for by means of a linear transformation a man starting from any point within the conic to walk to its infinitely far-off circumference at a uniform velocity will never reach it, much less then will he know what lies beyond.

He therefore concerned himself only with the points and angles within the absolute and saw that for every line the fundamental elements are real, but that for each pencil of rays they are imaginary, since no real tangent can be drawn from an interior point to the conic. He then put for C' the value $i/2$, so that the sum of the angles about a point is the same as in ordinary plane geometry.

This description of the absolute, that it is a real circle at infinity within whose circumference lie all real points, is exactly the same as the definition of Beltrami's limiting circle, and Beltrami's expression

$$\rho=\frac{R}{2}\log\frac{a+\sqrt{u^2+v^2}}{a-\sqrt{u^2+v^2}},$$

for the length of a geodesic line from the center of this circle is exactly the same as Klein's, if $C=R/2$. Consequently the propositions proved by Beltrami with respect to parallel lines, the angle of parallelism and trigonometrical ratios belong equally to the metrical geometry and may be solved by means of figures drawn in the plane of the absolute.

This geometry Klein called the Hyperbolic geometry and the spherical and Euclidean geometries he called Elliptic and Parabolic respectively, making the distinction between them depend upon whether the right line has two real, imaginary, or coincident points at infinity. He called the measure of distance of the Hyperbolic and Elliptic geometries, the general metrical determination, and that of the Parabolic, the special metrical determination. He remarked that the two may coincide at a point or in the neighborhood of a point, but that at points at a distance from the point of contact, the general metrical determination is greater or less than the special, according as to whether the fundamental conic is imaginary or real. He designated as measure of curvature the greatness of the respective gain and loss, and found that it is the same at every point, and that it is equal to $-1/4C^2$.

He regarded all the points and lines on the plane as the projections of lines and planes in space and the absolute as the section of a cone whose vertex lies at a determined point in space and which passes through the circle of infinity, and was able to prove that projective geometry can be completely developed, although absolutely free from the question of metrical determination. He thus showed that the hyperbolic geometry, since it has a real value for C is the geometry of surfaces with constant negative curvature and that the non-Euclidian geometry, the pseudospherical geometry and the hyperbolic geometry are essentially one and the same.

II

THE SURFACE OF CENTERS AND THE TRANSFORMATION OF PSEUDOSPHERICAL SURFACES.

1. The theorem that ∞^1 new pseudospherical surfaces may be derived from one that is known and the geometrical method for the determination of the new surfaces were derived by Bianchi[63] for a simple case only in 1879. In 1881 the theorem was developed analytically so as to apply to a more general case by Bäcklund[86] and was geometrically interpreted for this general case by Bianchi[99] in 1887.

In its generalized form the theorem may be stated as follows: if on a surface of constant negative curvature $-1/R^2$ a system of lines be chosen whose principal normals at every point make a constant angle $(\pi/2 - \sigma)$ with the normal to the surface at that point, and if tangent lines be drawn to these curves on each of which a constant length $R \cos \sigma$ is measured off, the extreme points of the constant lengths on these tangent lines will lie upon a second surface which has also constant negative curvature $-1/R^2$ and which can be completely determined when the first surface and σ are known.

This theorem as here written was not announced all at one time nor was it the work of a single man, but it is the result of the discoveries of many other theorems and it represents the labor of many men of different nationalities.

It is necessary in tracing the gradual development of the theory which lies beneath it, to go back to Kummer's[16] treatise on ray systems, for Bianchi[99] observed, that the tangent lines may be regarded as forming a congruence of right lines for which the two pseudospherical surfaces are the focal surfaces. Thus he pointed out that the whole theory of deriving new pseudospherical surfaces from a given one is dependent upon Kummer's theorems.

2. Kummer's paper entitled *The General Theory of Congruences of Right Lines* was published in 1860. His definitions and theorems which were afterward used by Bianchi in the development of this theory may be briefly stated as follows: suppose a system of rays to be composed of ∞^2 right lines, if all the rays of this system pass through a surface arbitrarily chosen as a surface of reference, each ray is determined by its direction cosines X, Y, Z and the coördinates x, y, z of its point of intersection with the surface with reference to a set of rectangular coördinates in space. If the linear element of the surface of refer-

ence be referred to curvilinear coördinates $u =$ a constant and $v =$ a constant, of parameters, the quantities x, y, z, X, Y, Z may be expressed as functions of u and v.

The abscissa of a point in space is its distance from the initial surface measured along the ray on which it lies.

Upon every ray of the system there lie five points which are of especial importance — the two limiting points, the two focal points and the middle point. The limiting points may be defined as follows: if to any ray (u, v) the lines of its shortest distance from all the rays of the system that are infinitely near it be drawn, the foot points on the ray of these lines of shortest distance will have, the one a maximum, the other a minimum abscissa, the foot points of all the other lines of shortest distance to the ray will lie between these two and they are called the limiting points. The lines of shortest distance drawn to the limiting points of a ray from the two rays infinitely near it are at right angles to each other, consequently the two planes drawn through the ray normal to these two lines of shortest distance respectively are perpendicular to each other.

The surface which is the locus of the limiting points with the maximum abscissæ and the surface which is the locus of the limiting points with the minimum abscissæ are called the principal surfaces.

The point on the ray which is half way between its limiting points is called its middle point.

The rays of the system themselves form two families of developable surfaces, real or imaginary, and two surfaces pass throught every ray. The curves of intersection of the two families of developable surfaces with the surface of reference are usually chosen for the parametric lines u and v respectively.

The surface which is the locus of the edges of regression of the developable surfaces of either of these families is called a focal surface. Every ray is tangent to both focal surfaces and its two points of contact with the two focal surfaces are called its focal points.

The curves along which the developable surfaces of the one family meet a focal surface are conjugate to the curves along which the developable surfaces of the other family meet that same focal surface.

In particular, Kummer proved that the condition that the right lines of a system shall become the normals to a surface, or rather to an infinite number of parallel surfaces, is that the focal surfaces shall coincide with the principal surfaces of the ray system and thus form the two nappes of the surface of centers or evolute surface of the parallel surfaces. A special case of the transformation of pseudospherical surfaces occurs when the tangents to the given surface become the normals to a series of parallel surfaces, but before taking up this subject it is necessary to make a study of the well known theorems which were derived by Weingarten, Beltrami and Dini with respect to the involute and evolute surfaces of surfaces with constant negative curvature.

3. Weingarten investigated a class of surfaces which are distinguished by the property that at every point one of their principal radii of curvature is a function of the other, and which are now called W-surfaces. Surfaces with constant negative curvature come under this heading and they are also connected with this class of surfaces by the fact that every pseudospherical surface is a nappe of the evolute of a W-surface.

Weingarten wrote two papers on these surfaces which appeared in Crelle's Journal. In the first,[17] published in 1861, he proved the theorem that the two nappes of the evolute surface of a surface whose principal radii of curvature R_1 and R_2 are bound together by a relation, $R_2 = \phi(R_1)$, are each applicable upon a surface of rotation, and that, if the first nappe corresponds to the lines of curvature $u =$ a constant of the involute surface along which the principal radius of curvature is denoted by R_1 and the second nappe corresponds to the lines of curvature $v =$ a constant of the involute surface along with the principal radius of curvature is represented by R_2, the expression for the linear elements of the first nappe assumes the form

$$ds_1^2 = dR_1^2 + e^{2\int \frac{dR_1}{R_1 - R_2}} du^2,$$

and that of the second nappe the form

$$ds_2^2 = dR_2^2 + e^{2\int \frac{dR_2}{R_2 - R_1}} dv^2.$$

He also demonstrated the converse of this theorem, that, if a surface is developable upon a surface of rotation, it may be considered as a nappe of the evolute of a surface whose radii of curvature are functionally related.

As a special illustration of the converse of this theorem, Weingarten supposed one of the nappes of surface of centers to be developable upon a catenoid, and proved that the relation which binds together the principal radii of curvature of its involute surface is

$$R_1 R_2 = -a^2 \qquad (a = \text{a constant}),$$

thus proving in an inverse fashion that a nappe of the evolute surface of a pseudospherical surface is applicable upon a catenoid.

In his second [20] paper on W-surfaces, published in 1863, he demonstrated a second theorem — that the spherical representation of the linear element of such a surface referred to its lines of curvature for parameters may be denoted by

$$ds^2 = \frac{1}{K^2} du^2 + \left(\frac{1}{\phi'(K)}\right)^2 dv^2,$$

where ϕ and K are functions of u and v of such a nature that they define the principal radii of curvature R_1, R_2 of the surface by means of the equations,

$$R_1 = \phi(K), \qquad R_2 = \phi(K) - K\phi'(K).$$

The expression for the linear element of the W-surface itself, when expressed in this notation, then becomes

$$ds^2 = \left(\frac{\phi(K)}{K}\right)^2 du^2 + \left(\frac{\phi(K) - K\phi'(K)}{\phi'(K)}\right)^2 dv^2.$$

Beltrami [26] in a series of articles on the application of analysis to geometry, published in 1865, proved both Weingarten's theorem and its converse and for the latter found that a ruled helicoidal surface forms a case of exception, for although this surface is applicable upon a surface of rotation, it cannot be the nappe of a surface of centers. He showed at this time that the curves on the nappe of a surface of centers which are enveloped by the normals to the involute surface are geodesic lines, he therefore remarked that "if the geodesic lines of an evolute surface become right lines the tangents to them at every point instead of filling all space reduce to a system of straight lines with a single parameter and are not sufficient to generate an orthogonal surface"; he discovered rather that, then, the geodesics themselves can generate a ruled surface which if it is applicable upon a surface of rotation is applicable upon the minimal surface of rotation, the catenoid, and is parallel to a series of pseudospherical surfaces instead of being a nappe of their evolute surface.

Dini [29] also investigated this case of exception to the converse of Weingarten's theorem in his paper on helicoidal surfaces, in the same year. He found that the ruled helicoid that is applicable upon a catenoid is a screw-surface generated by a right line that moves along a helix lying on a cylinder, making a right angle with the helix at every point and a constant angle with the cylinder, and that it may be regarded as the locus of the normals of another helicoidal surface upon which these same helices lie.

In the same treatise on the application of Analysis to Geometry [26] Beltrami made known several very important theorems concerning the surface of centers. He denoted a surface whose principal radii of curvature are functionally related by S, its lines of curvature by $u =$ a constant and $v =$ a constant, its principal radii of curvature and the two nappes of its evolute surface corresponding to those lines of curvature respectively by R_1, R_2 and S_1, S_2.

First, he demonstrated the general theorem that if two systems of curves, one of which is composed of geodesic lines, be conjugate to each other and if tangents be drawn to two of the geodesic lines that lie infinitely near each other at points a_1 and a_2 where they meet a curve of the other system, these tangents will meet at a point which is the center of geodesic curvature of a curve which passes through the point a_1, and is orthogonal to all the geodesic curves of the first system. Applying this proposition to the case of the nappes, S_1, S_2 of the evolute surface of a W-surface he obtained the results first, that, since the normals to the surface S taken along its lines of curvature $u =$ a constant touch the first

nappe of the surface of centers, S_1, along a family of geodesic lines, which are the evolutes of these lines of curvature and which may also be denoted by $u =$ a constant, and the normals to the surface taken along the lines of curvature $v =$ a constant, are tangent to this same nappe along curves which, as Kummer has shown,* are conjugate to the geodesic lines $u =$ a constant, S_2 is the locus of the centers of geodesic curvature of the orthogonal trajectories of the geodesic lines $u =$ a constant on S_1 and, conversely, the centers of geodesic curvature of the orthogonal trajectories of the geodesic lines corresponding to $v =$ a constant on S_2 lie on S_1; second, that the difference between the principal radii of curvature of the involute surface S at any point P is equal to the radius of geodesic curvature at a corresponding point of the orthogonal trajectory of the curves on either nappe which are the evolutes of the lines of curvature of the surface S, thirdly, that, when u_1 denotes the arc length of a curve in the nappe S_1 which goes through any point p corresponding to ρ and which is the evolute of a line of curvature $u =$ a constant in the surface S and when ρ denotes the radius of geodesic curvature for the point p of a curve orthogonal to u_1 and going through the point p, the principal radii of curvature R_1 and R_2 of the surface S at P are given by the equations

$$R_1 = u_1, \qquad R_2 = u_1 + \rho.\dagger$$

Beltrami's[26] direct contribution to the subject of pseudospherical surfaces at this time consisted in the determination of their evolute and involute surfaces. He first found the equation of relation connecting the principal radii of curvature R_1 and R_2 of any W-surface defined by the equation

$$R_2 = \phi(R_1)$$

and the principal radii of curvature R_1', R_2' of the surface of rotation on which is developable one of the nappes of its evolute surface. He wrote the equations for R_1' and R_2' in the usual form for the principal radii of curvature of the surface of rotation,

$$R_1' = -\frac{\sqrt{1-\left(\frac{dr_1}{du_1}\right)^2}}{\frac{d^2r_1}{du_1^2}}, \qquad R_2' = \frac{r_1}{\sqrt{1-\left(\frac{dr_1}{du_1}\right)^2}},$$

where $r_1 =$ the radius of a parallel circle and $u_1 =$ the arc of a meridian curve. Substituting in these equations the expressions for r_1 and dr/du_1,

$$r_1 = e^{\int \frac{du_1}{u_1-\phi(u_1)}}, \qquad \frac{dr_1}{du_1} = \frac{r_1}{u_1-\phi(u_1)}, \qquad \frac{d^2r_1}{du_1^2} = \frac{r_1\frac{d\phi(u_1)}{du_1}}{(u_1-\phi(u_1))^2},$$

* P. 47.

† Beltrami uses contrary signs for R_1 and u_1 in accordance with his definition of geodesic curvature.

derived from Weingarten's form for the linear element of the surface of rotation, he obtained the two equations

$$R_1' R_2' = \frac{[u_1 - \phi(u_1)]^2}{-\dfrac{d\phi(u_1)}{du_1}}, \qquad \frac{R_1'}{R_2'} = \frac{[u_1 - \phi(u_1)]^2 - r_1^2}{-r_1^2 \dfrac{d\phi(u_1)}{du_1}},$$

To find the evolute surface of a surface of constant negative curvature $-K^2$ he put in the equations for R_1' and R_2'.

$$R_1 R_2 = u_1 \phi(u_1) = -K^2$$

and found that they reduce to

$$R_1' + R_2' = 0,$$

the equation of a minimal surface, which shows that the nappes of the evolute surface of a pseudospherical surface are applicable upon a catenoid. To find the involute surface of a surface of constant negative curvature $-K^2$, he made in these same equations of relation the substitution $R_1' R_2' = -K^2$ and obtained the equation

$$R_2 - R_1 = K \frac{Ae^{-R_1/K} - Be^{R_1/K}}{Ae^{-R_1/K} + Be^{R_1/K}},$$

where A and B are constants of integration.

When neither A nor B is zero, he denoted their ratio by

$$\pm e^{2m}, \qquad (m = \text{arbitrary constant}),$$

so that

$$R_2 - R_1 = K \tanh\left(m - \frac{R_1}{K}\right),$$

$$R_2 - R_1 = K \operatorname{ctnh}\left(m - \frac{R_1}{K}\right),$$

according as the upper or lower sign of e^{2m} is taken. When either A or B is zero, his equation reduces to

$$R_2 - R_1 = \pm K,$$

which shows that the surface of rotation has parallels with constant geodesic curvature and must be a pseudosphere. Beltrami, therefore, announced the theorem "that the evolute surface of surfaces which have at every point the difference of their principal radii of curvature constant and equal to K is a surface of constant negative curvature $-1/K^2$."

This same theorem was proved in a more direct way by Enneper[35] in 1868. He used the subscripts (1) and (2) to denote the quantities on the first and second nappe respectively of an evolute surface, and from the equations for the coördinates of a point on each obtained all the coefficients of their two funda-

mental forms. He then found for the measure of curvature of each, K_1 and K_2, the expressions

$$K_1 = -\frac{1}{(R_1 - R_2)^2}\frac{dR_2}{dR_1}, \qquad K_2 = -\frac{1}{(R_1 - R_2)^2}\frac{dR_1}{dR_2}.$$

As a particular case he put

$$R_2 - R_1 = \text{a constant},$$

which necessitates that the curvature of each nappe is constant and negative.

4. The correspondence between the lines of curvature and that of the asymptotic curves on the two nappes of a surface of a W-surface was demonstrated by Ribaucour[50] in a paper read before the French Academy in 1872. He wrote the expression for the linear element of the initial surface referred to its lines of curvature as coördinates in the form

$$ds^2 = f^2du^2 + g^2dv^2,$$

and defined its principal radii of curvature by the expressions *

$$R_1 = \frac{f}{a}, \qquad R_2 = \frac{g}{b}. \qquad (a,\ b = \text{functions of } u \text{ and } v.)$$

He obtained for the lines of curvature on the first nappe the equation

$$\frac{dR_1}{(R_2 - R_1)bdv} = \frac{adu}{\frac{1}{f}\frac{\partial g}{\partial u}dv - \frac{1}{g}\frac{\partial f}{dv}du},$$

and for those on the second nappe the equation

$$\frac{dR_2}{(R_2 - R_1)adu} = \frac{bdv}{\frac{1}{f}\frac{\partial g}{\partial u}dv - \frac{1}{g}\frac{\partial f}{\partial v}du}.$$

Therefore, the condition that the lines on the nappes shall correspond is

$$dR_1 = dR_2$$

or that $R_2 - R_1 = \text{a constant}$. and both nappes have constant negative curvature, he wrote the equations for the asymptotic lines in the form

$$\frac{1}{f}\frac{\partial g}{\partial u}\cdot\frac{\partial R_1}{\partial v}dv^2 \mp \frac{1}{g}\frac{\partial f}{\partial v}\cdot\frac{\partial R_1}{\partial u}du^2 = 0,$$

and

$$\frac{1}{f}\frac{\partial g}{\partial u}\cdot\frac{\partial R_2}{\partial u}dv^2 \pm \frac{1}{g}\frac{\partial f}{\partial v}\cdot\frac{\partial R_2}{\partial u}du^2 = 0,$$

so that the condition for their correspondence is

$$\frac{\partial R_1}{\partial v}\cdot\frac{\partial R_2}{\partial u} - \frac{\partial R_1}{\partial u}\cdot\frac{\partial R_2}{\partial v} = 0,$$

or R_1 and R_2 are functions of each other.

* Bianchi,[160] § 64, 127, 128.

Ribaucour[43, 51] had at this time already communicated to the Academy a series of propositions in regard to the class of surfaces which he called "cyclic," and defined as a system of surfaces that have a family of circles for orthogonal trajectories. By so doing he virtually laid the foundation of the Transformation Theory that is, the theory of deriving an infinite number of surfaces of negative constant curvature from one that is known, but the connection between this theory and a cyclic system was not seen until Bäcklund * pointed it out ten years afterwards.

Ribaucour's propositions were, first, that if the family of circles are orthogonal to three surfaces, they will be orthogonal to an infinity of them; that these surfaces form part of a triply orthogonal system whose other two families are composed of the envelopes of spheres and that they are intimately connected with the theory of deformation; second, that in order to find all the trajectory surfaces when one (A) is determined, it is necessary to know a function Z on (A) which satisfies the partial differential equation,

$$\frac{\partial^2 Z}{\partial\rho\partial\rho_1} = \frac{1}{H}\frac{\partial H}{\partial\rho}\cdot\frac{\partial Z}{\partial\rho_1} + \frac{1}{H_1}\frac{\partial H_1}{\partial\rho_1}\cdot\frac{\partial Z}{\partial\rho},$$

where the linear element of the surface is denoted by

$$ds^2 = H^2 d\rho^2 + H_1^2 d\rho_1^2,$$

and that this equation is integrable at once when the lines of curvature for (A) are geodesic circles; third, that, in the special case, when the circles lie in the tangent plane of a given surface and have all the same constant radius, the surfaces orthogonal to these circles are all applicable upon a surface (A) which is itself applicable upon a pseudosphere; fourth, that if a system of curves are normal to family of surfaces that form part of an orthogonal system, the osculating circles of those curves will be normal to a family of surfaces that belongs to a cyclic system.

5. Proofs of these theorems were worked out ten years later by L. Bianchi[83, 88, 89, 90, 93, 94] in a series of elaborate treatises, but during these ten years the transformation theory itself was formally established, by means of which all pseudospherical surfaces may be obtained by quadrature alone when one is known. Bianchi[63, 69] gave the first conception of the theory in 1879 and published a more concise statement of his results in 1880. He regarded a pseudospherical surface as one of the nappes of the evolute of an unknown W-surface and proved that the second nappe, which he called the "complementary surface," may be one of an infinity of surfaces, each corresponding to a family of geodesic lines on the first nappe.

From the theorems of Weingarten and Beltrami he knew that the tangents

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* Page 59.

common to the two nappes touch them along geodesic lines, that both nappes are developable into surfaces of rotation and that either is the locus of the centers of geodesic curvature of the orthogonal trajectories of the geodesic lines on the other to which the normals of the involute surface are tangent. Choosing, then, a family of geodesic lines on the first nappe that became meridian curves when the surface is deformed into one of rotation, he obtained the second nappe by the following rule: "On each tangent to the geodesics of a system on the first nappe S_1 cut off a portion equal to the radius of geodesic curvature of the trajectory orthogonal to the geodesic at that point. The locus of the new extremes is the surface S_2 complementary to S_1."

Since every surface of constant negative curvature possesses three different kinds of systems of geodesic lines, those that go out from a real finite point, those that go out from a point at infinity and those that go out from an imaginary point, Bianchi discovered that the surface complementary to a pseudospherical surface is applicable upon a rotation surface of one of three different forms depending upon which kind of geodesic lines are selected on the original surface.

He wrote down the equations of the relation between the radii of curvature of the involute surface of surfaces of curvature $-1/a^2$ for the three cases, as they were given by Beltrami,*

$$R_1 - R_2 = a \tanh \frac{u + c}{a},$$

$$R_1 - R_2 = a,$$

$$R_1 - R_2 = a \operatorname{ctnh} \frac{u + c}{a}.$$

Substituting these values in Weingarten's formula for the linear element of the second nappe,

$$dS_2^2 = dR_2^2 + e^{\int \frac{dR_2}{R_2 - R_1}} dv^2.$$

he found the three corresponding expressions for the linear element to be

$$dS_2^2 = \tanh^4 \frac{R_1}{a} dR_1^2 + \operatorname{sech}^2 \frac{R_1}{a} dv^2.$$

$$dS_2^2 = dR_1^2 + e^{-2R_1/a} dv^2,$$

$$dS_2^2 = \operatorname{ctnh}^4 \frac{R_1}{a} dR_1^2 + \operatorname{csch}^2 \frac{R_1}{a} dv^2.$$

In regard to the profile curve he found that, although the equation of z is always the same

$$z = a \left(\log \tan \frac{\phi}{2} + \cos \phi \right),$$

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* Page 51.

where ϕ is the obtuse angle between the axis of rotation $z =$ a constant and the tangent to a meridan curve at any point, the equation for r, the radius of a parallel circle, is

$$r = \frac{a}{\sqrt{1 - a^2 k^2}} \sin \phi,$$

where $k^2 \gtreqqless 0$ for the three surfaces respectively.

These results led him to announce the theorem that "a surface complementary to a surface of constant negative curvature with respect to a system of geodesic lines which go out from a point on the surface is developable upon a rotation surface which has for its axis the asymptote and for its meridian curve, a curtailed tractrix, the ordinary one or an elongated one, according as the point of intersection is finite and real, at infinity, imaginary." "The first named curve," he said, "is none other than the orthogonal projection of the tractrix upon a plane which goes through the asymptote. On the other hand, the last named curve has the tractrix for its orthogonal projection."

He further observed that the deformed parallels of the surface of rotation upon which the complementary surface is applicable correspond to the deformed parallels of the surface of rotation into which the original surface is developable. He proceeded to find the equations for surfaces other than surfaces of rotation that are complementary to a pseudosphere with respect to a family of geodesic lines of each of the three kinds, and having found these equations he showed that the corresponding surfaces are applicable upon one of the three kinds of surfaces of rotation. Bianchi[69] also extended the application of this theorem to helicoidal surfaces, and found that there are also three kinds of helicoidal surfaces complementary to a pseudospherical helicoid corresponding to the three kinds of geodesic lines with reference to which they may be developed.

Since the surface complementary to a pseudospherical surface with respect to a system of geodesic lines going out from a point at infinity is developable upon a pseudosphere and has the same curvature as the original surface, and since there are ∞^1 systems of this sort on a surface of constant negative curvature, Bianchi remarked that from a pseudospherical surface S_1 an infinite number of new surfaces S_2 of the same curvature may be derived, and that from each surface S_2 an infinite number of new surfaces S_3, also with the same curvature, may be obtained in the same way as S_1 is obtained from S_2, provided that a family of geodesic lines on S_2 are known, and so on.

6. From Bianchi's surface that is complementary to a pseudospherical surface with reference to a family of geodesic lines going out from a point at infinity, Kuen, in 1879, derived by the repetition of Bianchi's operation the equations for a new pseudospherical surface which he classified as an Enneper surface. The paper in which Kuen[85] announced these results was referred to on page 26.

In the same year Lie[61, 63] developed Bianchi's theorem further. In a paper

published in the Archiv for Mathematik und Naturvidenskab he introduced a method for finding by means of a quadrature alone the geodesic lines of the surface of centers of a W-surface, and especially for the case when the surface of centers is composed of pseudospheres. "This problem for determining the geodesic lines," he observed, "is equivalent to determining the lines of curvature on the W-surface."

He supposed the W-surface to be referred to a system of curvilinear coördinates (x, y) and expressed one of its principal radii of curvature R_1 at a point and the coördinates x_1, y_1, z_1 of a point on the corresponding nappe S_1 of its evolute surface as functions of these parameters. He wrote the expression for the linear element on the nappe S_1 referred to a family of geodesic lines and their orthogonal trajectories as parameters and of curvature $-1/a^2$, in the usual form

$$ds_1^2 = dR_1^2 + e^{2R_1/a} dv^2.$$

From this he derived the equation for the geodesic lines

$$dv = e^{-R_1/a} \sqrt{(ds_1^2 - dR_1^2)} = e^{-R_1/a} \sqrt{dx_1^2 + dy_1^2 + dz_1^2 - dR_1^2}$$

and observed that the quantity under the sign of the radical is of the form

$$\{ X(x, y) dx + Y(x, y) dy \}^2,$$

where X and Y are functions of x and y only, so that he could at once obtain the integral of the equation containing an arbitrary constant. Therefore, if he had given a surface S_1 with curvature $-1/a^2$, he could bring its linear element in ∞^1 ways into the form

$$ds_1^2 = dR_1^2 + e^{2R_1/a} dv^2 \, *$$

referred to a family of geodesic lines going out from a point at infinity and their orthogonal trajectories, and considering this surface as the first nappe of a surface of centers he could derive an involute surface corresponding to one of those infinite systems of geodesic lines. From this involute surface he could obtain a second nappe S_2 of a surface of centers and by the method just given determine on it a family of geodesic lines and write its linear element in the form

$$ds_2^2 = dR_2 + e^{-2R_2/a} du^2.$$

A repetition with respect to S_2 of the operations performed on S_1 would then enable him to obtain a new set of surfaces S_3 and by the successive application of this same process he could derive ∞^∞ surfaces all with the same constant negative curvature $-1/a^2$. In actual practice he remarked "it is possible to go directly from one nappe to a second without stopping to obtain the involute surface."

It may here be remarked that several years later in 1888 Weingarten[101] developed another method for finding the lines of curvature on a W-surface and

* Page 12.

consequently the corresponding geodesic lines on its surface of centers, which, according to Darboux,[159 § 764], is "more precise but less direct" than that of Lie.

After having developed this method for determining the geodesic lines of a pseudospherical surface Lie next called attention to a method for transforming one surface into another that had been discovered by Bonnet many years before.

Bonnet[45, p. 76] had shown that every surface of constant mean curvature is applicable upon an infinite number of surfaces of the same sort and that such a surface is parallel to a surface of constant total curvature and obtainable from it by dilatation. Lie suggested therefore that if a parallel surface be derived from a pseudospherical surface and transformed into an infinity of new surfaces with constant mean curvature and each of these in its turn be transformed back into a pseudospherical surface, the result will be the same as if Bianchi's operation had been performed upon the original surface of constant negative curvature.

He showed moreover that the asymptotic curves of a surface of constant curvature may be found by a simple integration and that they correspond to the minimal lines of a parallel surface, a theorem which furnished the means of obtaining the equation of transformation as it is given by Darboux.*

Lie used for the linear element of the surface referred to its lines of curvature $u =$ a constant and $v =$ a constant the expression due to Weingarten[20]†

$$ds^2 = \left(\frac{\phi(K)}{K}\right)^2 du^2 + \left(\frac{\phi(K) - K\phi'(K)}{\phi'(K)}\right)^2 dv^2,$$

and for the asymptotic lines of the surface $u_1 =$ a constant, and $v_1 =$ a constant the corresponding expression

$$\frac{\phi}{K^2} du^2 + \frac{(\phi - K\phi')}{\phi'^2} dv^2 = 0.$$

This last equation he saw is integrable if

$$L\left(\frac{\phi}{K^2}\right) = \frac{\phi - K\phi'}{\phi'^2} \qquad (L = \text{constant}).$$

The general integral of this equation is

$$\phi^2 = AK^2 + LA^2,$$

and the total curvature of the corresponding surface is constant for

$$R_1R_2 = LA^2 \quad (R_1,\ R_2 = \text{principal radii of curvature}).$$

while a singular integral is

* DARBOUX,[159] § 775.

† Page 49.

$$\phi = \frac{i}{2\sqrt{L}} K^2,$$

and the mean curvature of the corresponding surface is constant for

$$\frac{1}{R_1} - \frac{1}{R_2} = 0 \quad (R_1, R_2 = \text{principal radii of curvature}).$$

He did not prove his theorem in detail nor give the equations of transformation deduced from Bonnet's theorem, but Darboux, [159, §775] in his celebrated work concerning surfaces, gives a simple proof for the correspondence between the asymptotic lines on the surface with constant curvature and the minimal lines on the surface with constant mean curvature and, then, denoting the linear element on the first surface referred to its asymptotic lines as parameters by

$$ds^2 = d\alpha^2 + 2 \cos \omega d\alpha d\beta + d\beta^2$$

and the linear element of a parallel minimal surface referred to its minimal lines as parameters by

$$ds^2 = 4e^{i\omega} d\alpha d\beta,$$

where 2ω is the angle between the asymptotic lines and also the angle between the minimal lines, he pointed out that, when either surface is transformed into a new surface of the same kind, the equations of transformation will be

$$\Omega(\alpha, \beta) = \omega\left(\frac{\alpha}{a}, a\beta\right),$$

where 2Ω is the angle between the asymptotic lines of the new surface with constant curvature or the angle between the minimal lines of the new surface with constant mean curvature and a is a constant.

The next year Lie [67] raised the question whether the surfaces obtainable from one that is known by Bianchi's method of transformation are all distinct from each other or whether a finite number of them are coincident, or as he expressed it, "whether those surfaces of constant curvature $1/a^2$, which are derived by the infinitely repeated successive application of Bianchi's operation from one that is given, must satisfy still other differential equations beside the equation

$$\frac{rt - s^2}{(1 + p^2 + q^2)^2} = \frac{1}{a^2}.$$

He answered this question in the negative and his method of proving his answer correct applies to surfaces of constant positive curvature as well as to those of constant negative curvature.

He began his demonstration by writing down the equations which represent known characteristic properties of the nappes of an evolute surface with constant negative curvature

$$(x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2 = a^2,$$
$$p(x - x_1) + q(y - y_1) - (z - z_1) = 0\,,$$
$$p_1(x - x_1) + q_1(y - y_1) - (z - z_1) = 0\,,$$
$$p_1 p + q_1 q + 1 = 0\,,$$

where (x, y, z, p, q) and $(x_1, y_1, z_1, p_1, q_1)$ determine an element on the first and second nappes respectively.

The first equation shows that the distance between corresponding points on the nappes is constant, the last three, that tangent planes to the nappes at corresponding points meet at right angles along a common tangential line. In this way he showed, as Bäcklund[86] remarked, that the surface on whose tangent planes lie the circles with constant radius of a Ribaucour cyclic system and the family of surfaces normal to the circles are identical respectively with Bianchi's initial surface and the infinite number of its complementary surfaces, for these equations express analytically the fact that a system of surfaces,

$$z_1 = f(x_1, y_1)$$

are orthogonal at the point (x, y, z) to a family of circles with constant radius a and with their centers lying on the lines of curvature of a surface

$$z = f(x, y).$$

Lie called his initial system of equations the equations of transformation, and since he had four of them from which to eliminate the five variables, he saw that to every element (x, y, z, p, q) there corresponds an infinity of elements $(x_1, y_1, z_1, p_1, q_1)$, so that to the ∞^2 elements that go to make up the original surface there corresponds ∞^3 new elements, and he proved that these ∞^3 new elements can form ∞^1 surfaces when the curvature is constant.* By applying his equations for transformation he was thus able to obtain ∞^1 surfaces ϕ_1 from one surface F and from these derived surfaces ϕ_1, ∞^2 new surfaces F_2, among which may be the first surface F. By repeating successively this operation he finally obtained ∞^{2m} surfaces F and ∞^{2m+1} surfaces ϕ, for the surfaces of the one class are finitely distinct from those of the other, but he had still to decide whether he could derive all the pseudospherical surfaces in this way, or only a limited number of them.

He considered two surfaces, F and F_2, which differ so little that the one may be deformed into the other by an infinitesimal transformation. By carefully working out the equations for this infinitesimal transformation he found three different equations for δp and δq, the increments of p and q, for determining the way in which an element x, y, z, p, q passes into its next adjacent position. He then assumed that this element could not go over into all the new elements

* Cf. page 65.

but only into a certain number of them which form a locus defined by the equation

$$f(x, y, z, p, q) = 0.$$

He saw that this locus must be deformed into itself by the same operations which transform the elements infinitesimally and that, therefore, it must satisfy three equations of condition, one corresponding to each of the three different pairs of value of δp and δq. But from these same equations of condition he found that the partial derivatives of the first order of f with respect to each of the five variables vanish independently, that consequently the locus f cannot exist, but that each element passes over in all the new elements and the given surface is deformed by the equations of transformation into ∞^3 new surfaces.

In like manner he found that the given surface can satisfy no partial differential equation of the second or third order and accordingly may be transformed into ∞^5 or ∞^9 new surfaces, but he could not arrive at any general result by this method. He[166] next turned to the consideration of a strip on the given surface formed by an aggregation of successive elements and, therefore, transformable into ∞^1 new strips. He proved by the actual application of the equations for a Bianchi transformation that, if the curve C, formed by the points of all the surface elements along a strip is an asymptotic curve it may be deformed into ∞^1 new asymptotic curves K, that the arc length of each new curve K is equal to the corresponding arc length of C and that the curvature $1/R_1$ of each new curve is related to the curvature $1/R$ of C by the equation

$$\frac{a}{R} = \frac{a}{R_1} - 2 \sin \nu.$$

where ν is the angle that a line joining a point on the one curve to a corresponding point on the other curve makes with the tangent to either curve at the point where the curve is met by the line. He derived ∞^2 new asymptotic curves C_2 from the curves K; by a third repetition of the operation he obtained ∞^4 new curves K_2 and so on, so that the problem as in the case of surfaces resolves itself into the question, is there any limit to the number of asymptotic curves that are thus derived? Reasoning in the same way as for the infinitesimal transformation of surfaces, he found that the number of asymptotic curves that can be derived from one that is known will be reduced, only, if these derived curves can satisfy an ordinary differential equation. Denoting a/R by v and a/R_1 by v_1 he wrote the equation connecting these values in the form

$$v = v_1 - 2 \sin \nu$$

and the equation for ν in the form

$$a \frac{d\nu}{ds} = - v_1 + \sin \nu.$$

He then proved that such an equation as

$$f(v, v') = 0 \qquad \left(v' = a\frac{dv}{ds}\right)$$

cannot exist, so that any asymptotic curve corresponding to $v =$ a constant can be transformed into at least ∞^1 new asymptotic curves. He proved that there is no relation between v and its derivatives of the second or third order with respect to s nor, indeed, between v and its derivatives of any order for on account of the form of the equations for the increment of v, δv and δv^2 he could write down by analogy the equation for δv^n and then show that the one for δv^{n+1} is exactly similar. He thus showed that there is no limit to the number of asymptotic lines that can be obtained from a given one by the equations of transformation.

He then turned back to the case of the surfaces and, by means of his new results, increased the number of surfaces that can be derived from a known one which passes through two intersecting asymptotic curves from ∞^9 to ∞^∞ thus establishing his theorem.*

Lie proved that there is not only a correspondence between the asymptotic lines on a transformed surface with those on the initial surface but also one between their lines of curvature, for since, according to Dini's discovery, the asymptotic lines of a surface divide it into lozenges, a net-work of lozenges on one surface S is transformable into a net-work of lozenges on each of the derived surfaces, and the lines of curvature which are their diagonals pass over into lines of curvature.

During these same years from 1879 to 1882 while Bianchi and Lie were making their important investigations on the method of obtaining new surfaces, of constant curvature from a given one, Bäcklund [70, 75] was publishing the results of his studies on the transformation of surfaces in successive volumes of the Mathematische Annalen and the discoveries of Bianchi and Lie were made just at a time when Bäcklund could use them as examples to illustrate his theorems.

Among other propositions Bäcklund [70] considered the question whether two surfaces may be transformed into each other when the relation between them is of such a nature that it is defined by four arbitrary partial differential equations of the first order.

He denoted the two surfaces by

$$z = \phi(x, y) \qquad \text{and} \qquad z' = f(x', y'),$$

and using p, q, r, s, t to denote the partial derivatives of the first and second order of z with respect to x and y as is customary, and p', q', r', s', t' to denote

* Bianchi, [160] § 247.

the partial derivatives of the first and second order of z' with respect to x' and y' he wrote the four partial differential equations in the form

$$\begin{aligned} &F_1(x, y, z, p, q, x', y', z', p', q') = 0. \\ &F_2(\qquad\qquad\qquad\qquad\qquad\quad) = 0. \\ &F_3(\qquad\qquad\qquad\qquad\qquad\quad) = 0. \\ &F_4(\qquad\qquad\qquad\qquad\qquad\quad) = 0. \end{aligned}$$

He then proceeded to find under what condition the surface whose equation is

$$z = \phi(x, y)$$

may be transformed into the surface whose equation is

$$z' = f(x', y')$$

by means of these equations, he first substituted in the equations $F_1 = 0$ and $F_2 = 0$ the values of z, p, q expressed as functions of x and y. He then solved the resulting equations for x and y expressing them in terms of the accented variables only. By means of these results he could, by making the proper substitutions, reduce the last two equations

$$F_3 = 0 \text{ and } F_4 = 0$$

to equations containing x', y', z', p', q' only, in which case he denoted them by

$$F_3' = 0 \text{ and } F_4' = 0.$$

He could then obtain the function z' by means of these equations provided that they are compatible. The equation of condition which must be satisfied by z' when the two equations

$$F_3' = 0, \qquad F_4' = 0$$

are compatible may be obtained by first taking the total derivatives of each of these equations, which are also equal to zero, then solving the resulting equations for dp' and dq' so that

$$\left\{\begin{aligned} \begin{vmatrix} \frac{\partial F_3'}{\partial q'}, & \frac{\partial F_3'}{\partial p'} \\ \frac{\partial F_4'}{\partial q'}, & \frac{\partial F_4'}{\partial p'} \end{vmatrix} dp' &= \begin{vmatrix} \frac{\partial F_3'}{\partial x'} + p'\frac{\partial F_3'}{\partial z'}, & \frac{\partial F_3'}{\partial q'} \\ \frac{\partial F_4'}{\partial x'} + p'\frac{\partial F_4'}{\partial z'}, & \frac{\partial F_4'}{\partial q'} \end{vmatrix} dx' + \begin{vmatrix} \frac{\partial F_3'}{\partial y'} + q'\frac{\partial F_3'}{\partial z'}, & \frac{\partial F_3'}{\partial q'} \\ \frac{\partial F_4'}{\partial y'} + q'\frac{\partial F_4'}{\partial z'}, & \frac{\partial F_4'}{\partial q'} \end{vmatrix} dy' \\ \begin{vmatrix} \frac{\partial F_3'}{\partial q'}, & \frac{\partial F_3'}{\partial p'} \\ \frac{\partial F_4'}{\partial q'}, & \frac{\partial F_4'}{\partial p'} \end{vmatrix} dq' &= \begin{vmatrix} \frac{\partial F_3'}{\partial x'} + p'\frac{\partial F_3'}{\partial z'}, & \frac{\partial F_3'}{\partial p'} \\ \frac{\partial F_4'}{\partial x'} + p'\frac{\partial F_4'}{\partial z'}, & \frac{\partial F_4'}{\partial p'} \end{vmatrix} dx' + \begin{vmatrix} \frac{\partial F_3'}{\partial y'} + q'\frac{\partial F_3'}{\partial z'}, & \frac{\partial F_3'}{\partial p'} \\ \frac{\partial F_4'}{\partial y'} + q'\frac{\partial F_4'}{\partial z'}, & \frac{\partial F_4'}{\partial p'} \end{vmatrix} dy' \end{aligned}\right.$$

and finally setting the coefficient of dy' in the first of these last equations equal

to the coefficient of dx' in the second equation from which results the equation

$$\left(\frac{dF'_3}{dx'}\right)\frac{\partial F'_4}{\partial p'}-\left(\frac{dF'_4}{dx'}\right)\frac{\partial F'_3}{\partial p'}+\left(\frac{dF'_3}{dy'}\right)\frac{\partial F'_4}{\partial q'}-\left(\frac{dF'_4}{dy'}\right)\frac{\partial F'_3}{\partial q'}=0.$$

where

$$\left(\frac{dF'_i}{dx'}\right)=\frac{\partial F'_i}{\partial x'}+p'\frac{\partial F'_i}{\partial z'}+r'\frac{\partial F'_i}{\partial p'}+s'\frac{\partial F'_i}{\partial q'}$$

$$\left(\frac{dF'_i}{dy'}\right)=\frac{\partial F'_i}{\partial y'}+q'\frac{\partial F'_i}{\partial z'}+s'\frac{\partial F'_i}{\partial p'}+t'\frac{\partial F'_i}{\partial q'} \qquad (i=3,4).$$

This equation he represented by the bracket $[F'_3F'_4]^*_{z'x'p'}=0$ and when instead of F'_3 and F'_4 he introduced their equivalent values in terms of the unaccented variable the equation became

$$[F'_3F'_4]=[F_3F_4]_{z'x'p'}+\frac{dF_3}{dx}[xF_4]+\frac{dF_3}{dy}[yF_4]+\frac{dF_4}{dx}[F_3x]$$
$$+\frac{dF_4}{dy}[F_3y]+\left[\frac{dF_3}{dx}\frac{dF_4}{dy}-\frac{dF_3}{dy}\frac{dF_4}{dx}\right][xy]=0$$

or finally

$$[F'_3F'_4]=(34)[F_1F_2]_{z'x'p'}+(42)[F_1F_3]_{z'x'p'}+(23)[F_1F_4]_{z'x'p'}$$
$$+(12)[F_3F_4]_{z'x'p'}+(13)[F_4F_2]_{z'x'p'}+(14)[F_2F_3]_{z'x'p'}=0,$$

where

$$(mn)=\left(\frac{dF_m}{dx}\right)\left(\frac{dF_n}{dy}\right)-\left(\frac{dF_m}{dy}\right)\left(\frac{dF_n}{dx}\right) \qquad (n,m=1,2,3,4).$$

He had, therefore, three equations

$$F'_3=0, \qquad F'_4=0, \qquad [F'_3F'_4]=0,$$

containing the accented variables only, which will determine a surface $z'=f(x',y')$ and only one surface, provided that these equations are in involution, a condition which he represented in the usual manner by the equations

$$[F'_3(F'_3F'_4)]=0, \qquad [F'_4(F'_3F'_4)]=0.$$

He moreover showed that the function $z'=f(x',y')$ will satisfy two partial differential equations of the third order obtained by eliminating x', y', z', p', q' from the four equations of transformation and from the equations of condition $[F'_3F'_4]=0$. He remarked that an exception to this theorem occurs when z' does not appear in the equation

$$[F'_3F'_4]=0$$

and that then ∞^1 surfaces $z'=f(x',y')$ will correspond to one surface $z=\phi(x,y)$,

* $z'x'p'$ written after the bracket signifies that F_i is differentiated with respect to the accented variables only.

for, in that case, instead of two partial differential equations of the third order for z there will be one single partial differential equation of the second order, and if an integral of this equation be substituted for z in the equation of transformation, the quantities x', y', p', q' can be expressed in terms of x, y, z' so that the function z' will be determined by an equation of the form

$$dz' = A(x, y, z')dx + B(x, y, z')dy.$$

The integral of this last equation will contain an arbitrary constant which proves the theorem that there are ∞^1 surfaces $z' = f(x', y')$ corresponding to one surface $z = \phi(x, y)$.

Bäcklund[75] saw that a surface transformation of this nature occurs in Bianchi's problem for deriving a surface complementary to a known surface of constant negative curvature. He considered the two surfaces defined by

$$z = \phi(x, y) \qquad z' = f(x', y')$$

as the two nappes of the evolute surface whose radii of curvature are connected by the relation

$$R_2 - R_1 = a.$$

The two relations existing between the two nappes, that the distance between corresponding points is a constant a and that their tangent planes at corresponding points must meet at a right angle along the common tangent, gave him his four equations of transformation

$$F_1 = p(x' - x) + q(y' - y) - (z' - z) = 0$$
$$F_2 = p'(x' - x) + q'(y' - y) - (z' - z) = 0$$
$$F_3 = 1 + pp' + qq' = 0$$
$$F_4 = (x - x')^2 + (y - y')^2 + (z - z')^2 - a^2 = 0.$$

and his equation of condition took the form

$$(rt - s^2) + a^2(1 + p^2 + q^2)^2 = 0,$$

since the expressions $[F_3 F_4]_{z'x'p'}$, $[F_2 F_3]_{z'x'p'}$, $[F_1 F_2]_{z'x'p'}$, $[F_1 F_4]_{z'x'p'}$ all become equal to zero. He saw from this equation and from a similar one for z, since the equations of transformation are symmetrical with respect to the accented and unaccented variables, that both surfaces are of constant negative curvature $-1/a^2$ and that since z' does not appear in the equation of condition that there correspond an infinity of surfaces $z' = f(x', y')$ to every surface $z = \phi(x, y)$.

In 1884 Bäcklund[86] wrote an important paper that deals exclusively with pseudospherical surfaces. In this paper, published in the Lund's Uni-

versitets Årsskrift and entitled "Concerning Surfaces with Constant Negative Curvature," he first reviewed the contributions made to the theory of the transformation of the pseudospherical surfaces by Bianchi, Ribaucour and Lie, pointing out the close connection between the theories of Bianchi and those of Ribaucour, he then extended Bianchi's theorem to fit a more general case, namely, when the given surface and the derived surface are not the nappes of an evolute surface but are so related to each other that planes tangent to them at corresponding points cut each other at a constant angle, but not at right angles, and the distance between two corresponding points is constant. He expressed this condition by leaving the first three equations of transformation unaltered and writing $F_4 = 0$ in the form

$$F_4 = 1 + pp' + qq' - K(\sqrt{1 + p^2 + q^2})(\sqrt{1 + p'^2 + q'^2}) = 0$$

where K is the cosine of the angle formed by the two tangent planes and is a constant.

He then found that the equation for z becomes

$$rt - s^2 = -\frac{1 - K^2}{a^2}(1 + p^2 + q^2)^2,$$

and that a like one exists for z', so that in the general case also both surfaces have constant negative curvature. By putting for $(1 - K^2)/a^2$ a constant $1/m^2$ and letting a and K vary, he obtained an infinity of equations of transformation for surfaces only whose curvature is $-1/m^2$ and in particular those for Bianchi's complementary transformation when $K = 0$ and $m = a$.

He made a complete study of this general method of transformation. First he remarked that the set of equations

$$z = f(x),\ y = \phi(x), \qquad p = \psi(x), \qquad q = \frac{f'(x) - \psi(x)}{\phi'(x)}$$

determine a curve on the surface together with the direction of the tangent plane to the surface along that curve for successive values of x, that is, they determine a strip of the surface. Then recalling Cauchy's theories he observed that if x, y and z are the coördinates of an arbitrary point in the strip and x_0, y_0, z_0 the coördinates of its initial point, a surface passing through this strip and satisfying a known differential equation may be defined by a convergent Taylor's series in terms of $(x - x_0)$, $(y - y_0)$ where the singular points of the surface are not considered, thus

$$z - z_0 = p_0(x - x_0) + q_0(y - y_0) + \tfrac{1}{2}\{r_0(x - x_0)^2 + 2s_0(x - x_0)(y - y_0) + t_0(y - y_0)^2\}$$
$$+ \frac{1}{3!}\{u_0(x - x_0)^3 + 3v_0(x - x_0)^2(y - y_0) + 3w_0(x - x_0)(y - y_0)^2 + \omega_0(y - y_0)^3\}$$
$$+ \cdots.$$

For the surfaces under discussion he obtained the values of the coefficients r, s, etc., from the equations

$$dp = rdx + sdy, \qquad dq = sdx + tdy,$$

$$rt - s^2 = -\frac{1}{m^2}(1 + p^2 + q^2)^2,$$

so that t is determined by the equation

$$t = \frac{dq^2 - \frac{1}{m^2}(1 + p^2 + q^2)^2 dx^2}{dpdx + dqdy}.$$

In general only one surface can be found passing through the strip, but when the value of t is indeterminate, that is, when

$$dpdx + dqdy = 0 \qquad \text{and} \qquad dq^2 - \frac{1}{m^2}(1 + p^2 + q^2)^2 dx^2 = 0,$$

Bäcklund saw that there are an infinity of surfaces having contact of the first order along the lines defined by those equations and that these lines are the characteristic curves of the integral surface. Since, the first of these equations shows that each curve may have for its plane of osculation at every point the tangent plane to the surface on which it lies at that point and the second equation shows that the torsion of the curve is constant, Bäcklund thus proved that the characteristic curves of surfaces of negative constant curvature are asymptotic curves.

He, then, demonstrated geometrically that the guiding curve of every strip r', derived from a strip r on the original surface S, by means of the set of equations of transformation satisfies a partial differential equation of the Riccati type and that, consequently, every strip r' corresponds to a solution of such an equation.

This final result may then be stated as follows: If the surface S is known, all the surfaces S' may be derived from it, for every strip r on S passes over into an infinity of strips r', one on each surface S', by means of the equations of transformation, therefore each derived surface S' corresponds to the solution of a Riccati equation and when one such surface is determined, all the others may be found by quadrature alone, since that is the only operation required to obtain all the solutions of a Riccati equation when one is known. Bäcklund has proved geometrically that the asymptotic curves on the new surfaces S' are the deformed asymptotic curves of the original surface S and that they also satisfy an equation of the Riccati form.

In 1883 and 1884, Bianchi [88, 89, 90, 93, 99] published his investigations of Ribaucour's propositions concerning a system of surfaces which have a family of ∞^2 circles for their orthogonal trajectories. Bäcklund [86] had already observed that,

when the circles all have the same constant radius and lie in the tangent planes of a known surface, this known surface and the surfaces orthogonal to the circles are identical with a pseudosphere and the ∞^1 surfaces derived from it by means of a complementary transformation with respect to a family of geodesic lines on it that go out from a point at infinity. Bianchi gave an exact proof of the identity of the two families of surfaces by establishing the theorem that a surface orthogonal to a family of ∞^2 circles, can be regarded as the nappe of the evolute surface of a W-surface, provided that the line of intersection of the plane of every circle with the planes tangent to the orthogonal surface at its point of contact with that circle envelopes geodesic lines on the surface and by showing that those enveloped curves are geodesic lines when the radius of the circles is always the same. Bianchi's construction of a cyclic system of surfaces is as follows:

Let S_1* be a surface orthogonal to a family of circles. Let these circles lie on the tangent planes of a second surface S_2, and let the points of tangency of those planes with the surface S_2 be the centers of the circles. Let $u =$ a constant and $v =$ a constant denote the lines of curvature of the surface S_2 and let θ be the angle that a radius of the circle, mn, drawn to meet an orthogonal surface S_1 at m makes at its center n with the line of curvature $v =$ a constant passing through that point. The radius of the circle mn being tangent to the orthogonal surface S_1 must lie in the tangent plane at m and is the line of intersection of the plane of the circle with the corresponding tangent plane of the orthogonal surface S_1. Let u' and v' be the lines of curvature of S_1. Let ϕ be the angle which this line of intersection makes with the tangent to the line of curvature of S_1, $v' =$ a constant, at the point of contact m. When the surface S_2 is regarded as known, each orthogonal surface S_1 corresponds to a value of θ. When an orthogonal surface S_1 is regarded as known, each circle is determined by its radius and the value of ϕ to which it corresponds.

Bianchi [88] denoted by $\Phi =$ a constant, the curves on the orthogonal surface S_1 which are the orthogonal trajectories of the curves on that surface that are enveloped by the lines of intersection of the planes of the circles with the corresponding tangent planes of the surface S_2. When the linear element of this surface S_1 referred to its lines of curvature as parameters assumes the form

$$dS_1^2 = Edu^2 + Gdv^2,$$

he showed that Φ must be a solution of the differential equation

$$\frac{\partial^2\Phi}{\partial u\partial v} = \frac{\partial\Phi}{\partial u}\cdot\frac{\partial\Phi}{\partial v} + \frac{1}{\sqrt{E}}\frac{\partial\sqrt{E}}{\partial v}\cdot\frac{\partial\Phi}{\partial u} + \frac{1}{\sqrt{G}}\frac{\partial\sqrt{G}}{\partial u}\cdot\frac{\partial\Phi}{\partial v},$$

which, if $\log Z$ is written in place of Φ, reduces to an equation for Z identical with that given by Ribaucour.

* DARBOUX,[159] § 804; BIANCHI,[160] § 179.

He also proved Ribaucour's statement, that it is necessary to be able to integrate this equation in order to find all the cyclical systems of which the surface S_1 forms a part, for he derived for R and ϕ, the functions by which a circle is determined, the following expressions

$$\frac{1}{R^2} = \Delta_1 \Phi,^* \qquad \cos\phi = R\frac{1}{\sqrt{E}}\frac{\partial \Phi}{\partial u}, \qquad \sin\phi = R\frac{1}{\sqrt{G}}\frac{\partial \Phi}{\partial v},$$

which can be found when the surface S_2 and a value of ϕ are known.

From his equations for expressing the condition that the circles are orthogonal to a surface S_1, Bianchi was able to show that when the circles have all the same constant radius R, the surface S_1 as well as the surface S_2, on whose tangent planes the circles lie, will both have constant negative curvature $-1/R^2$. In that case he saw that

$$\Delta_1 \Phi = \text{a constant},$$

or that $\Phi = $ a constant are geodesic parallel circles and that the curves enveloped by the lines of intersection of the planes of the circles with the corresponding tangent planes of the orthogonal surface S_1 and which are the orthogonal trajectories of $\Phi = $ a constant will be geodesic lines. Moreover, he found that the geodesic curvature, $1/\rho$, of the curves $\Phi = $ a constant is equal to $1/R$ so that when R is constant they are the deformed horicycles of the pseudosphere on which the surface S_1 of curvature $-1/R^2$ is applicable. The fact that the curves on the surface S_1 that are enveloped by the lines of intersection of the planes of the circles with the corresponding tangent planes to the surface S_1 are geodesic lines was the only condition Beltrami required in order to prove that the surfaces S_1 and S_2 form the nappes of an evolute surface, for these lines of intersection, being tangent to the surface S_1 along geodesic lines, may be regarded as the rays of a normal congruence of which the surfaces S_1 and S_2 are the focal surfaces.

It is not necessary to give in detail the equations and theorems by means of which Bianchi proved Ribaucour's propositions, that if a system of ∞^2 circles are orthogonal to three surfaces they will be orthogonal to ∞^1 surfaces and the theorems relating to the triply orthogonal systems to which these surfaces belong, but it is important to consider a proof given by Darboux[80] in 1883 for the establishment of the theorem regarding the existence of this triply orthogonal system, for in that connection Darboux[159, p. 426] developed for the first time the now well known set of equations for performing a complementary transformation. He regarded as known the surface S_2 of curvature -1 on whose tangent planes lie the circles of the system. He chose the lines of curvature

* $\Delta_1 \Phi = \frac{1}{G}\left(\frac{\partial \Phi}{\partial v}\right)^2 + \frac{1}{E}\left(\frac{\partial \Phi}{\partial u}\right)^2$, Bianchi,[160] §§ 35, 86.

of this surface for its lines of reference and wrote the linear element in the form

$$ds^2 = \cos^2 \omega du^2 + \sin^2 \omega dv^2,$$

where ω satisfies the characteristic equation

$$\frac{\partial^2 \omega}{\partial u^2} - \frac{\partial^2 \omega}{\partial v^2} = \sin \omega \cos \omega$$

and is half the angle between the asymptotic curves, $u + v$ and $u - v$.* He referred to both Ribaucour and Bianchi, and using the notation of the latter, denoted by θ, the angle that the line $v = a$ constant makes with the radius mn of a circle in a plane tangent to S_2 at m. He saw that the coördinates of n relative to the tetrahedron at m are

$$\cos \theta, \qquad \sin \theta, \qquad 0,$$

and expressing the condition that this line mn whose direction cosines with respect to the tetrahedron's axes are

$$-\sin \theta d\theta + \cos \omega du - \left(\frac{\partial \omega}{\partial v} du + \frac{\partial \omega}{\partial u} \right) \sin \theta,$$

$$\cos \theta d\theta + \sin \omega dv + \left(\frac{\partial \omega}{\partial v} du + \frac{\partial \omega}{\partial u} dv \right) \cos \theta.$$

$$\cos \omega \sin \theta dv - \sin \omega \cos \theta du$$

should be perpendicular to the tangent to the circle at n, whose direction cosines are

$$-\sin \theta, \qquad \cos \theta, \qquad 0,$$

he obtained the equation

$$d\theta + \frac{\partial \omega}{\partial v} du + \frac{\partial \omega}{\partial u} dv - \sin \theta \cos \omega du + \sin \omega \cos \theta dv = 0,$$

and consequently the equations

$$\frac{\partial \theta}{\partial u} + \frac{\partial \omega}{\partial v} = \sin \theta \cos \omega, \qquad \frac{\partial \theta}{\partial v} + \frac{\partial \omega}{\partial u} = -\cos \theta \sin \omega,$$

which are consistent when the equation for ω is satisfied.

He further observed that each solution for θ contains an arbitrary constant α so that there may be an infinity of surfaces S_1. He considered θ as a function of u, v and α, and wrote for $d\theta$,

$$\frac{\partial \theta}{\partial u} du + \frac{\partial \theta}{\partial v} dv + \frac{\partial \theta}{\partial \alpha} d\alpha$$

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* P. 28.

in the expressions for the displacement of m, and brought them to the form

$$\cos\theta\,(\cos\omega\cos\theta du + \sin\omega\sin\theta dv) - \sin\theta\frac{\partial\theta}{\partial\alpha}\,d\alpha,$$

$$\sin\theta\,(\cos\omega\cos\theta du + \sin\omega\sin\theta dv) + \cos\theta\frac{\partial\theta}{\partial\alpha}\,d\alpha,$$

$$\cos\omega\sin\theta dv - \sin\omega\cos\theta du,$$

which give for the displacements of n in space

$$ds^2 = \cos^2\theta du^2 + \sin^2\theta dv^2 + \left(\frac{\partial\theta}{\partial\alpha}\right)^2 da,$$

a formula which demonstrates the existence of the triply orthogonal system.

Bianchi[99] obtained a like set of equations for representing a Bäcklund transformation. Employing the same expression for the linear element of the initial surface referred to its lines of curvature as Darboux had used,

$$ds^2 = \cos^2\omega du^2 + \sin^2\omega dv^2,$$

and denoting by σ the complement of the angle between the tangent planes at corresponding points of this surface and a derived surface, he first wrote these equations in the form:

$$\frac{\partial\theta}{\partial u} + \frac{\partial\omega}{\partial v} = \frac{\sin\theta\cos\omega + \sin\sigma\cos\theta\sin\omega}{R\cos\sigma},$$

$$\frac{\partial\theta}{\partial v} + \frac{\partial\omega}{\partial u} = -\frac{\cos\theta\sin\omega + \sin\sigma\sin\theta\cos\omega}{R\cos\sigma}$$

and, by using asymptotic lines on the initial surface for parameters instead of lines of curvature, reduced them to the simpler form

$$\frac{\partial(\theta-\omega)}{\partial u_1} = \frac{1+\sin\sigma}{R\cos\sigma}\sin(\theta+\omega),$$

$$(u_1 = u - v, \quad v_1 = u + v)$$

$$\frac{\partial(\theta+\omega)}{\partial v_1} = \frac{1-\sin\sigma}{R\cos\sigma}\sin(\theta-\omega),$$

He saw that these equations are compatible if the curvature of the initial surface is constant and negative, and that they form a Riccati equation for $\tan\theta/2$ such as Bäcklund had obtained previously.

Bianchi represented a Bäcklund transformation by B_σ and his own, or the complementary transformation, when $\sigma = 0$, by B_0. Later he denoted a Lie transformation in which, retaining the previous notation,

$$\theta(u, v) = \omega\left(au, \frac{v}{a}\right)^*$$

* Page 58.

by L, and wrote

$$a = \frac{1 + \sin\sigma}{\cos\sigma} \quad \text{and} \quad \frac{1}{a} = \frac{1 - \sin\sigma}{\cos\sigma}.$$

He[160, p. 460] found that a Bäcklund transformation is a combination of a Lie transformation and a complementary transformation, or that

$$B_\sigma = L_\sigma B_0 L_\sigma^{-1},$$

where the negative exponent denotes an inverse operation.

Bäcklund's idea of a constant angle, not a right angle, between the tangent planes at corresponding points on the two surfaces S_1 and S_2 that are deformable into each other, apparently recalled to Bianchi's mind Kummer's treatise on ray systems. He[99] asked himself the question, can two pseudospherical surfaces developable into each other by a Bäcklund transformation form the two focal surfaces of a congruence of right lines?

Now Kummer has shown that if γ denotes the angle between the tangent planes to the focal surfaces respectively that pass through a common ray,

$$\sin\gamma = \frac{\delta}{d},$$

where $2d$ is the distance between the limiting points of the ray and 2δ is the distance between its foci. Bianchi,[99] observing that when δ and γ are constant, d must be constant also, considered a ray system subject to these conditions and let S_1 and S_2 denote its two focal surfaces, that is, S_1 and S_2 denote the two focal surfaces of a ray system which is characterized by the property, that the distance between the limiting points on every ray is equal to a constant R and the distance between the points of intersection of every ray with the focal surfaces is also a constant and equal to $R \cos\sigma$, where $(\pi/2 - \sigma)$ is the constant angle between planes tangent at corresponding points to the focal surfaces.

He[99] took one of the focal surfaces for the surface of reference and calling the curves on it which are enveloped by the rays, "caustics," chose them and their orthogonal trajectories for the coördinate lines u and v. He was then able by means of Kummer's equations for the abscissæ of points on the focal surface and for the abscissæ of limiting points on the rays to prove, first, that "the inclination of the principal normal of any caustic of the focal surface to the tangent plane to the surface is equal to the mutual inclination of the tangent planes to the focal surfaces which pass through the tangent to the caustic," and, second, that the two focal surfaces have their curvature constant, negative and equal to the reciprocal of the square of the distance of the limiting points R. He gave the name pseudospherical congruence to a ray system whose focal surfaces are pseudospherical surfaces and proved that to any pseudospherical

surface S_2 there belongs a single infinity of pseudospherical congruences for which S_2 is the common focal surface.

In order to answer his original question and demonstrate that two focal surfaces of a pseudospherical congruence are developable into each other by a Bäcklund transformation, he regarded one of these surfaces S_2 as known and defined by the equation

$$\frac{\partial^2 2\omega}{\partial u \partial v} = \cos 2\omega,$$

where $2\omega =$ the angle between its asymptotic curves, $u =$ a constant and $v =$ a constant, and denoted by θ the angle which a ray, touching this surface at any point P and going out in the direction of the other focal surface S_2 which is to be determined, makes with a line of curvature of S_1 passing through P. He then proved that S_1 and S_2 are connected by the operations for a Bäcklund transformation so that the one surface S_1 may be transformed into the other surface S_2 by this method, that the angle θ has the same significance for S_1 as ω has for S_2 and that the linear element of S_2 may be written

$$ds^2 = \sin^2\theta du^2 + \cos^2\theta dv^2,$$

when referred to its lines of curvature.

This theorem enabled him to construct a geometrical method of performing Bäcklund's transformation analogous to his own early method for deriving a complementary surface from one arbitrarily chosen, for, he had only to select on a pseudospherical surface of any curvature $-1/R^2$ a family of curves whose principal normals make a constant angle $(\pi/2 - \sigma)$ with the tangent planes to the surface and cut off on the tangents to these curves a constant length equal to $R \cos\sigma$, then the locus of the extremes will be the required surface.*

As a special illustration of a Bäcklund transformation, he investigated the case when the surface derived by that method from a known surface degenerates into a straight line. Representing the linear element on the derived surface referred to its lines of curvature by

$$ds^2 = \sin^2\theta du^2 + \cos^2\theta dv^2,$$

he expressed the condition for a straight line by putting

$$\sin\theta = 0 \qquad \text{or} \qquad \cos\theta = 0,$$

and using the first expression, reduced his former equations of transformation to

$$\frac{\partial\omega}{\partial u} = -\frac{\sin\omega}{R\cos\sigma}, \quad \frac{\partial\omega}{\partial v} = \frac{\tan\sigma\sin\omega}{R}$$

whose common solution is

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* Page 47.

$$\tan\frac{\omega}{2} = e^{\frac{v\sin\sigma - u}{R\cos\sigma}+c}.$$

If the second expression is used this expression becomes

$$\tan\left(\frac{\pi}{4}+\frac{\omega}{2}\right) = e^{\frac{u\sin\sigma - v}{R\cos\sigma}+c},$$

but in either case, as Bianchi said, "The initial pseudospherical surface corresponding to this value of ω is Dini's screw surface," that is, the surface derived from a pseudospherical helicoidal surface by a Bäcklund transformation is a straight line.

From the study of individual pseudospherical surfaces, Bianchi [89, 90] turned to the investigation of triply orthogonal systems that contain a family of surfaces with constant negative curvature. He [138] then proceeded to the consideration of such a system as a unit and the transformation of one system into another by a complementary or Bäcklund transformation, so that if the pseudospherical surfaces undergo either transformation they still belong to a triply orthogonal system. He divided these systems of surfaces into two classes according to whether the pseudospherical surfaces of the one family have each a different radius of curvature or each the same radius of curvature.

Bianchi did not devote much time to the study of surfaces of the first class, but wrote several papers on those of the second class, which he called a Weingarten system after their discoverer, for immediately after Bianchi had published his proofs of Ribaucour's theorems on the cyclic system in 1884, Weingarten wrote to him suggesting the possibility of deriving from an initial surface of constant curvature a whole family of surfaces each with the same constant curvature and each lying at an infinitely short distance from the one next to it, while the family to which they belong form part of a triply orthogonal system.

In this way the transformation theory was rapidly but continuously developed without any overlapping of results or coincidence of discoveries. Each mathematician added his completed proposition to the theory and then stood aside while the next one took it up and developed it farther.

Bianchi's conception of the deformation of one pseudospherical surface into others of the same kind was at the beginning purely geometrical. He derived a complementary surface by cutting off equal lengths on the tangents of a family of geodesic lines going out from a point at infinity on the original surface; and his analytic work simply interpreted this geometrical notion. Next Lie showed that the geodesics, consequently the new surfaces, are obtainable by quadrature alone, derived ∞^{∞} surfaces instead of ∞^1 from the given one and wrote down the initial equations of transformation in a form that made it evident they defined equally Ribaucour's cyclic system and Bianchi's complementary transformation. Bianchi proved completely the identity of these two theorems and

in demonstrating Ribaucour's propositions, introduced the angle θ that a radius of a circle makes with a line of curvature of an orthogonal surface; Darboux made use of this angle in developing a new set of equations that are more practical for actual transformation than Lie's; Bäcklund introduced a new transformation, a generalization of Bianchi in which the tangent planes at corresponding points of the two surfaces meet at a constant angle but not at a right angle. Bianchi gave a geometrical method for performing Bäcklund's transformation and practically completed the subject.

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